Counting Fiedler pencils using diagrams

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88th GAMM Annual Meeting
Weimar, March 2017

Fiedler linearizations I

$$\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ A_0 & A_1 & A_2 & A_3 \end{bmatrix} - x \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -A_4 \end{bmatrix}$$

is a linearization (=same eigenvalues and multiplicities) of the matrix polynomial

$$A_3x^3 + A_2x^2 + A_1x + A_0 \in \mathbb{R}[x]^{n\times n}.$$

Fiedler linearizations I

$$\underbrace{\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ A_0 & A_1 & A_2 & A_3 \end{bmatrix}}_{\begin{bmatrix} I & & & \\ & I & & \\ & & & I \\ & & I & A_3 \end{bmatrix}\begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix}\begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix}\begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}} - x \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -A_4 \end{bmatrix}$$

is a linearization (=same eigenvalues and multiplicities) of the matrix polynomial

$$A_3x^3 + A_2x^2 + A_1x + A_0 \in \mathbb{R}[x]^{n\times n}.$$

Fiedler linearizations II

. . . and so is any product of the same factors in different order [Fiedler, '03, Antoniou-Vologiannidis '04]

$$
\underbrace{\begin{bmatrix} & & A_0 & \\ I & & A_1 & \\ & & & I \\ & I & A_2 & A_3 \end{bmatrix}}_{\begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}\begin{bmatrix} I & & & \\ & I & & \\ & & & I \\ & & I & A_3 \end{bmatrix}\begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix}\begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix}} - x \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -A_4 \end{bmatrix}.
$$

Fiedler linearizations III

...and there are more with factors of the same kind. [Vologiannidis-Antoniou '11]

$$\begin{bmatrix} & & & I \\ & & A_0 & A_1 \\ A_0 & & A_1 & A_2 \\ & I & A_2 & A_3 \end{bmatrix} - x \begin{bmatrix} & & I & \\ A_0 & & A_1 & \\ & I & A_2 & \\ & & & -A_4 \end{bmatrix}$$

$$= \begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & & I \\ & & I & A_3 \end{bmatrix} \begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix} \begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}$$

$$- x \begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -A_4 \end{bmatrix} \begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}.$$

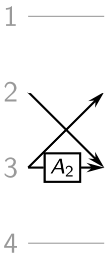
Plan

Finding new linearizations and using them numerically is a very active research area.

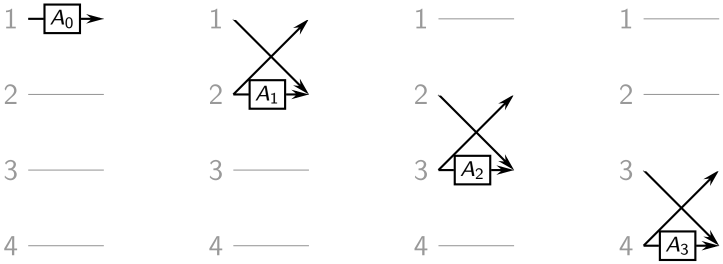
Special interest in finding structured (symmetric, palindromic. . .) linearizations of structured matrix polynomials.
[Mackey-Mackey-Mehl-Mehrmann '06]

In this talk: we study Fiedler linearizations using signal-flow graphs / computational diagrams.

$$F_2 = \begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix} =$$



Diagrams for elementary factors

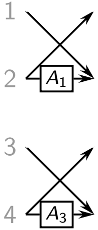


$$\underbrace{\begin{bmatrix} A_0 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}}_{F_0} \quad \underbrace{\begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix}}_{F_1} \quad \underbrace{\begin{bmatrix} I & & & \\ & & I & \\ & I & A_2 & \\ & & & I \end{bmatrix}}_{F_2} \quad \underbrace{\begin{bmatrix} I & & & \\ & I & & \\ & & & I \\ & & I & A_3 \end{bmatrix}}_{F_3}$$

Commuting factors

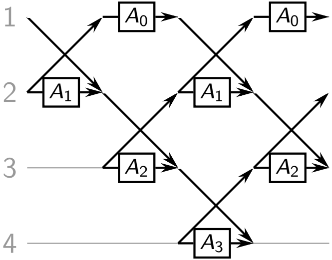
$$F_3 F_1 = F_1 F_3$$

$$\begin{bmatrix} I & & & \\ & I & & \\ & & & I \\ & & I & A_3 \end{bmatrix} \begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix} = \begin{bmatrix} & I & & \\ I & A_1 & & \\ & & I & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I & & \\ & & & I \\ & & I & A_3 \end{bmatrix} =$$



More complex diagrams

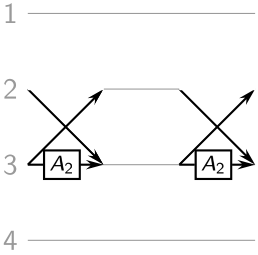
$$\begin{bmatrix} & & & I \\ & & A_0 & A_1 \\ A_0 & & A_1 & A_2 \\ & I & A_2 & A_3 \end{bmatrix} =$$



Operation-freeness

Desirable property of our linearizations: no operations needed.

Does not hold for all Fiedler products:

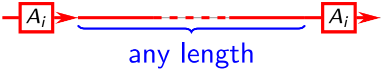


$$= \begin{bmatrix} I & & & \\ & I & A_2 & \\ & A_2 & A_2^2 + 1 & \\ & & & I \end{bmatrix}$$

Theorem ([Vologiannidis-Antoniou '11])

A Fiedler product is operation-free unless there are two factors F_k without a F_{k+1} in-between.

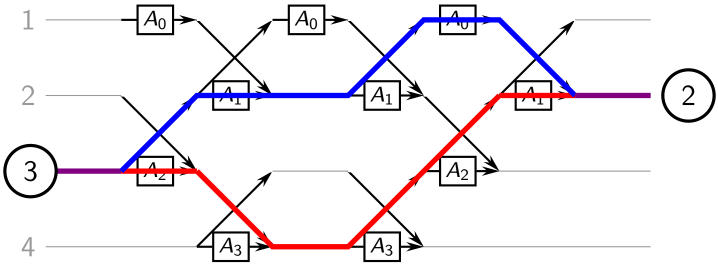
'Forbidden shape':



Sketch of a (diagram-based) proof

Nontrivial part: take a non-operation-free product:

$$\begin{bmatrix} 0 & A_0^2 & A_0A_1 & 0 \\ I & A_1 & A_2 & A_3 \\ A_2 & {\color{blue}A_1A_0} + {\color{red}A_2A_1} & A_1^2 + A_2^2 + A_0 & A_2A_3 \\ A_3 & A_3A_1 & A_3A_2 & A_3^2 + I \end{bmatrix}$$



Sketch of a (diagram-based) proof

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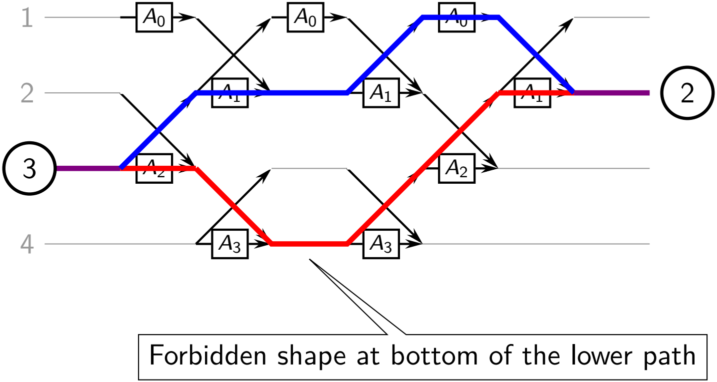
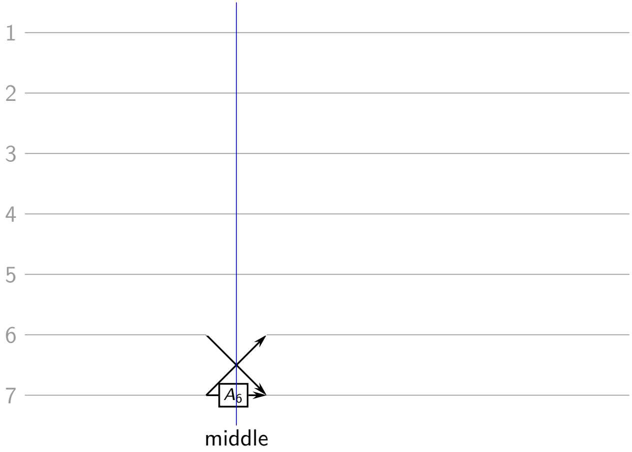


The Middle Standard Form

Rule: draw each block as close to the middle as possible.

$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 \color{red}{F_6} F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



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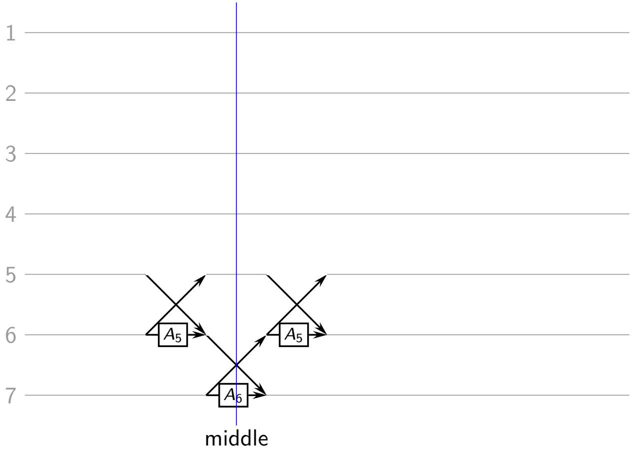
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



The Middle Standard Form

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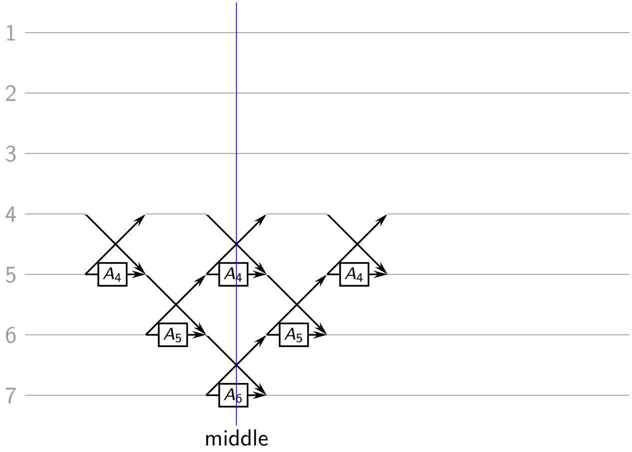
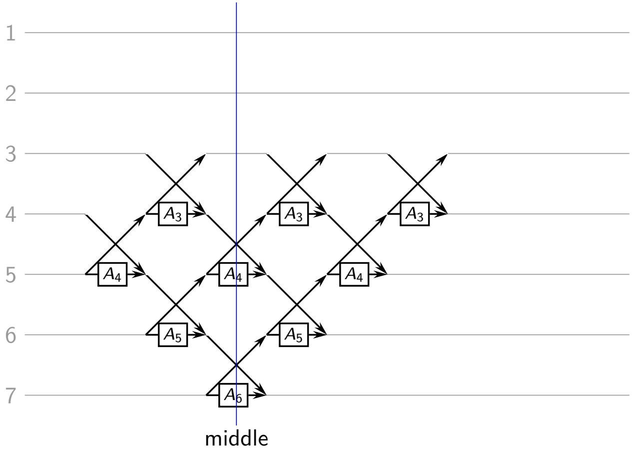


The Middle Standard Form

Rule: draw each block as close to the middle as possible.

$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



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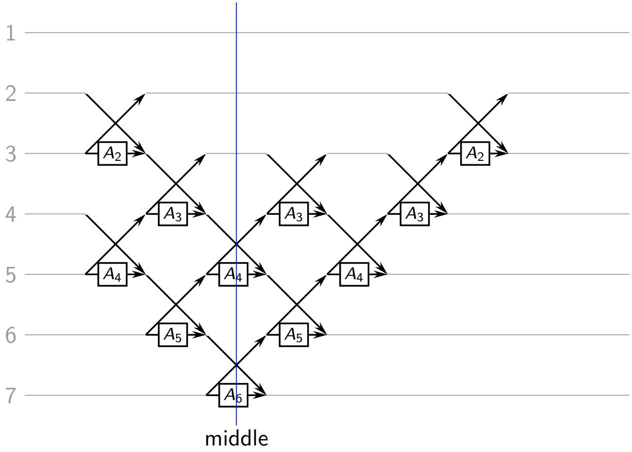
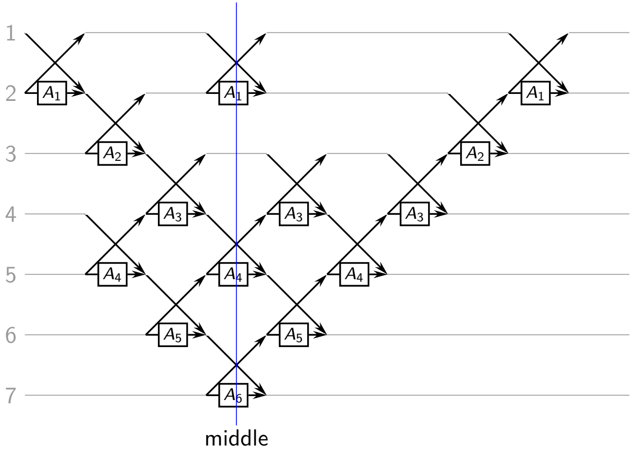
$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



The Middle Standard Form

Rule: draw each block as close to the middle as possible.

$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



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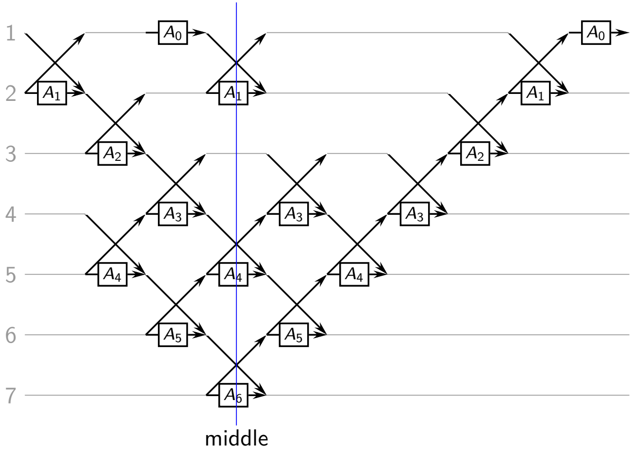
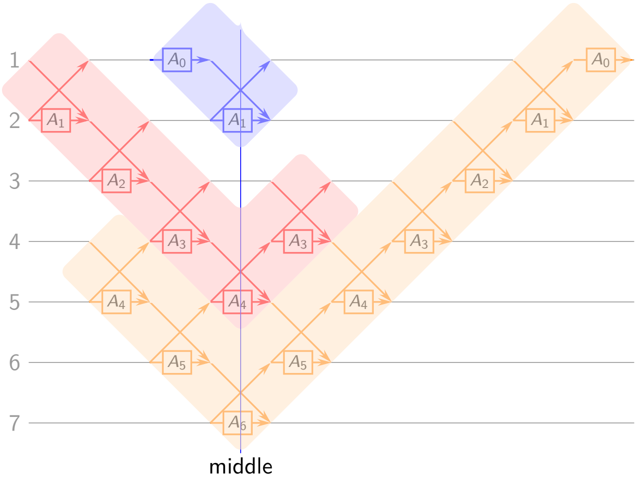


The Middle Standard Form

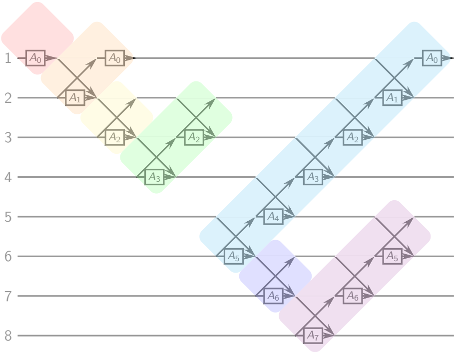
Rule: draw each block as close to the middle as possible.

$$F_4 F_5 F_1 F_2 F_3 F_0 F_1 F_4 F_6 F_3 F_5 F_4 F_3 F_2 F_1 F_0$$



Other standard forms

Row standard form: blocks as far left as possible (inside a triangular frame)



Naturally arranged in diagonals.

Column standard form: blocks as far right as possible.

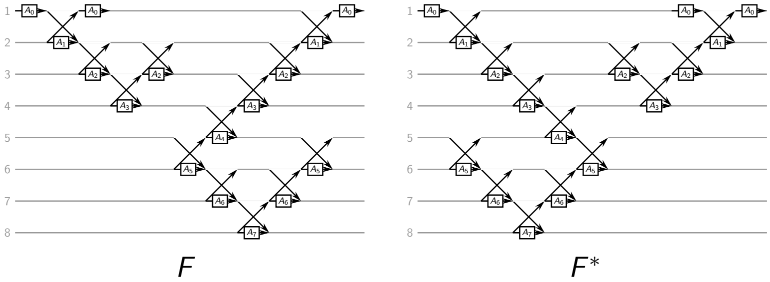
Corollary ([Vologiannidis-Antoniou '11])

For a degree-d $A(x)$, there are $(d+1)!$ operation-free products.

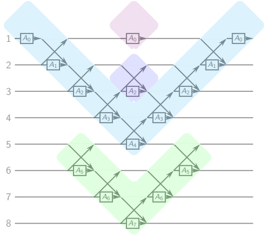
Symmetric matrix polynomials

If $A(x)$ is symmetric (i.e., $A_i = A_i^*$ for all i), then

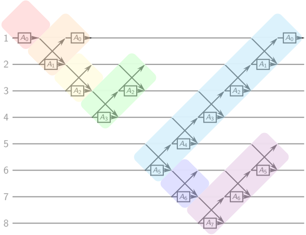
- the blocks $F_i = F_i^*$ are symmetric;
- a diagram for F^* is obtained by flipping horizontally one for F.



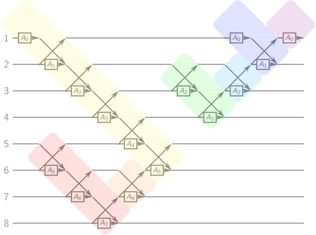
The MSF reveals symmetry



Middle standard form



Row standard form



Column standard form

Symmetric Fiedler products

Theorem

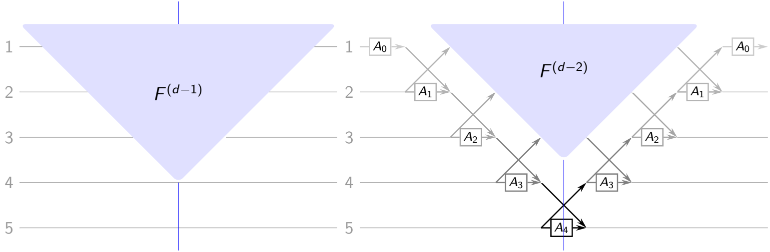
$F = F^* \iff$ *The MSF of F has left-right symmetry.*

We can count symmetric Fiedler products through their MSF:

Theorem

Let Σ_d be the number of different operation-free symmetric Fiedler products constructed from symmetric $A(x)$ of degree d. Then,

$$\Sigma_d = \Sigma_{d-1} + d\,\Sigma_{d-2}, \quad \Sigma_1 = 2,\ \Sigma_2 = 4.$$



Looking up this sequence

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2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152, 568504, 2390480 Search Hints

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Search: **seq:2,4,10,26,76,232,764,2620,9496,35696,140152,568504,2390480**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000085 Number of self-inverse permutations on n letters, also known as involutions; number of Young tableaux with n cells. +20 263
(Formerly M1221 N0469)

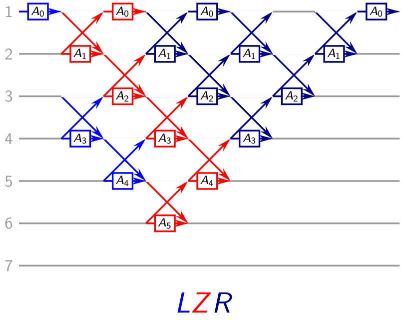
1, 1, **2**, **4**, **10**, **26**, **76**, **232**, **764**, **2620**, **9496**, **35696**, **140152**, **568504**, **2390480**, 10349536, 46206736, 211799312, 997313824, 4809701440, 23758664096, 119952692896, 618884638912, 3257843882624, 17492190577600, 95680443760576, 532985208200576, 3020676745975552 (list; graph; refs; listen; history; text; internal format)

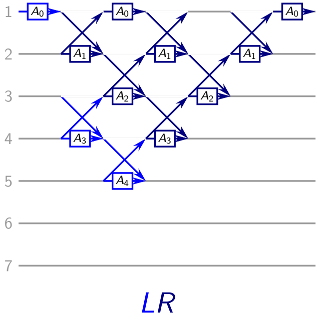
OFFSET 0,3

COMMENTS Getu (1991) says these numbers are also known as "telephone numbers". - N. J. A. Sloane, Nov 23 2015
a(n) is also the number of n X n symmetric permutation matrices.
a(n) is also the number of matchings (Hosoya index) in the complete graph K(n). - Ola Veshta (olaveshta(AT)my-deja.com), Mar 25 2001.
Equivalently, this is the number of graphs on n labeled nodes with degrees at most 1. - Don Knuth, Mar 31 2008
a(n) is also the sum of the degrees of the irreducible representations of the symmetric group S_n - Avi Peretz (njk(AT)netvision.net.il), Apr 01 2001
a(n) is the number of partitions of a set of n distinguishable elements into sets of size 1 and 2. - Karol A. Penson, Apr 22 2003

Infix pairs

Pairs of op-free products with and without an infix ‘zig-zag’ path.

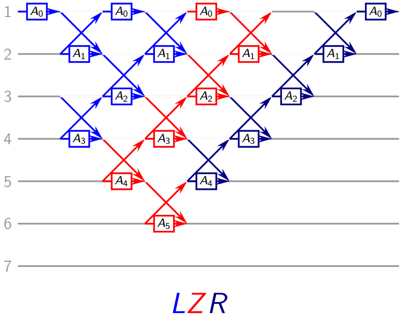


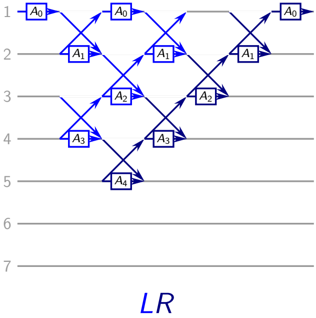


Basic building block for Fiedler-type linearizations.

Infix pairs

Pairs of op-free products with and without an infix 'zig-zag' path.

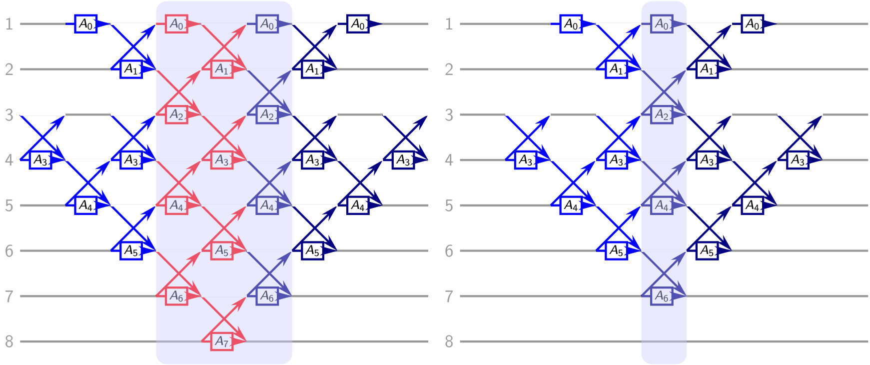




Basic building block for Fiedler-type linearizations.

Warning: coloring not uniquely determined.

Standard form for symmetric infix pairs



Theorem

In a symmetric infix pair (up to recoloring / equivalence): the 3 middle columns are completely filled and contain entirely the zig-zag path.

Similar (but more complicated) form in [Bueno-Curlett-Furtado '06].

Counting infix pairs

Theorem

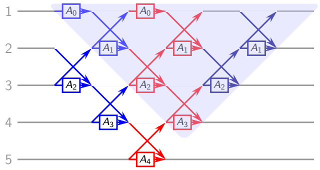
The number of symmetric op-free infix pairs (for $A(x)$ of degree d) is

$$(d-1)!! = (d-1)(d-3)(d-5)\cdots(2 \text{ or } 1).$$

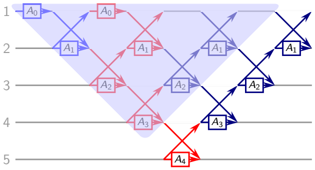
Theorem

The number Π_d of op-free infix pairs (for $A(x)$ of degree d) satisfies

$$\Pi_d = 2d\Pi_{d-1} - (d-1)^2\Pi_{d-2}, \quad \Pi_1 = 1, \Pi_2 = 3.$$



Left extension



Right extension

Number of infix pairs

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1, 3, 14, 85, 626, 5387, 52882, 582149, 7094234, 94730611, 1374650042 Search Hints

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A005189 Number of n-term 2-sided generalized Fibonacci sequences. +20 1
(Formerly M2976)

1, 1, **1**, **3**, **14**, **85**, **626**, **5387**, **52882**, **582149**, **7094234**, **94730611**, **1374650042**, 21529197077, 361809517954, 6492232196699, 123852300381986, 2502521367966277, 53379537613065002, 1198434678728086019, 28245547605034208074, 697186985180529270101 (list; graph; refs; listen; history; text; internal format)

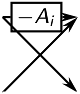
OFFSET 0,4

REFERENCES N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

LINKS N. J. A. Sloane, Table of n, a(n) for n = 0..250
C. Banderier, H.-K. Hwang, V. Ravelomanana and V. Zacharovas, Analysis of an exhaustive search algorithm in random graphs and the n^{c logn}-asymptotics, SIAM J. Discrete Math., 28(1), 342-371. (30 pages), DOI:10.1137/130916357. - From N. J. A. Sloane, Dec 23 2012
Peter C. Fishburn, Peter C. Marcus-Roberts, Fred S. Roberts, Unique finite difference measurement, SIAM J. Discrete Math. 1 (1988), no. 3, 334-354.
P. C. Fishburn, A. M. Odlyzko and F. S. Roberts, 2-sided generalized Fibonacci sequences, Fib. Quart., 27 (1989), 352-361.

FORMULA If n <= 2 then a(n) = 1 otherwise a(n) = 2*(n-1)*a(n-1)-(n-2)^2*a(n-2).
E.g.f.: (e*Ei(1/(x-1)) - e*Ei(-1)-1)/(e^(x/(x-1))*(x-1)), where Ei is the

Finally, linearizations

Fiedler pencil with repetitions = infix pair on the top + an analogous pair of inverse Fiedler matrices 

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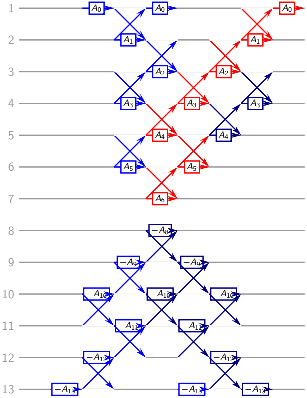
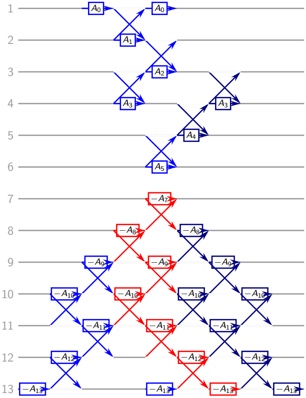
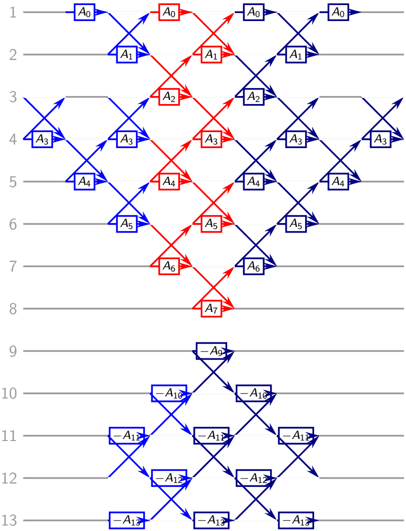


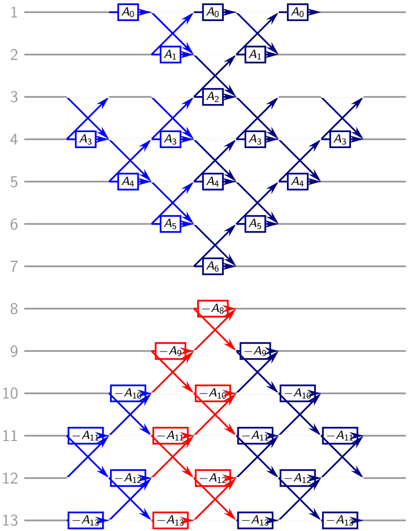


$$L(Z^+ - xZ^-)R.$$

Symmetric linearizations

Symmetric top pair + symmetric bottom pair.





Counting linearizations

Theorem

For a degree-d matrix polynomial $A(x)$, there are

$$\sum_{h=1}^{d} \Pi_h \Pi_{d+1-h}$$

different Fiedler pencils with repetitions.

Theorem

For a degree-d symmetric matrix polynomial $A(x)$, there are

$$\sum_{h=1}^{d} (h-1)!!(d-h)!!$$

different symmetric Fiedler pencils with repetitions.

Not on OEIS

THE ON–LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

1, 6, 37, 254, 1958, 16910 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,6,37,254,1958,16910**

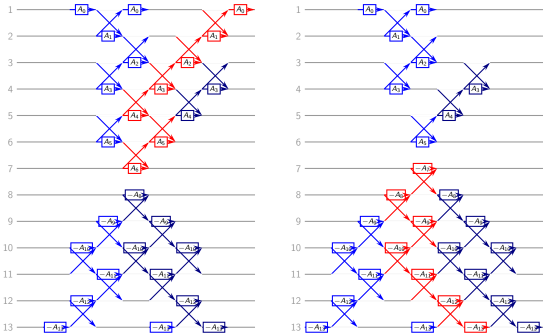
Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using
the form provided and it will (probably) be added to the OEIS!
Include a brief description and if possible enough terms to fill
3 lines on the screen. We need at least 4 terms.

(Anti-)palindromic pencils

$A(x)$ antipalindromic matrix polynomial, i.e., $A_i = -A^*_{d-i}$.

Are there antipalindromic Fiedler pencils with repetitions, i.e., $F - F^*x$?

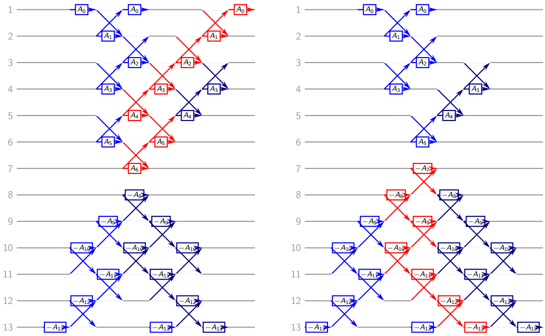


(Anti-)palindromic pencils

$A(x)$ antipalindromic matrix polynomial, i.e., $A_i = -A^*_{d-i}$.

Are there antipalindromic Fiedler pencils with repetitions, i.e., $F - F^*x$?

No (apart from the trivial case $d = 1$).



J-antipalindromic pencils

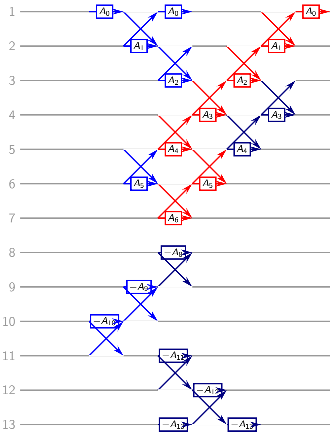
However, for odd d, one can have $J(F - F^*x)$ antipalindromic [Bueno-Furtado '12], with

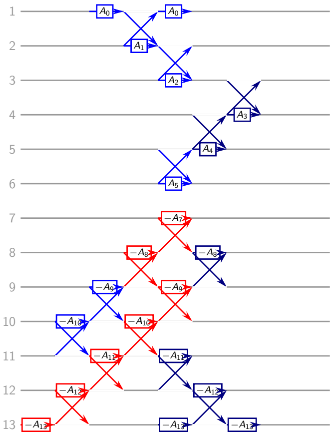
$$J = \begin{bmatrix} & & & I_n \\ & & I_n & \\ & \iddots & & \\ I_n & & & \end{bmatrix}.$$

Remark: for even d, there are no general constructions for palindromic linearizations (not only Fiedler ones). [Mackey-Mackey-Mehl-Mehrmann '11]

J-antipalindromic Fiedler pencils

J = up/down flip. A pencil $F - xG$ is J-antipalindromic if G can be obtained by F with an up-down and a left-right flip. E.g.:





J-antipalindromic Fiedler pencils

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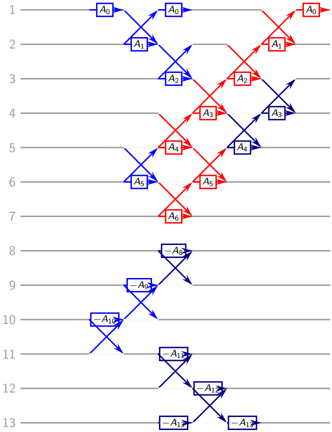

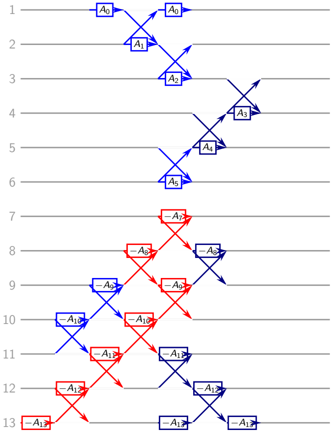

There are $\Pi_{\frac{d+1}{2}}$ of them.

Conclusions

- Diagrams are a powerful tool to understand, characterize and count Fiedler-type pencils.
- New proofs and results, standard and special forms.
- Some issues ignored in this talk, e.g., different diagrams $\leftrightarrow$ different matrices?

Symmetries in the matrix polynomial

$\Updownarrow$

Symmetries in the spectrum

$\Updownarrow$

Symmetries in the diagram

Conclusions

- Diagrams are a powerful tool to understand, characterize and count Fiedler-type pencils.
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Symmetries in the matrix polynomial

$\Updownarrow$

Symmetries in the spectrum

$\Updownarrow$

Symmetries in the diagram

Thanks for your attention!
Questions?