Multivariate time series estimation via projections and matrix equations

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Moving average models

Moving average models [Lütkepohl '06 book, Tsay '14 book]

$\{u_t \in \mathbb{R}^n\}$ Gaussian independent $N(0, I)$

$$y_t = A_0 u_t + A_1 u_{t-1} \qquad MA(1)$$

$$y_t = A_0 u_t + A_1 u_{t-1} + A_2 u_{t-2} + \cdots + A_q u_{t-q} \qquad MA(q)$$

(A_i real square matrices, y_t, u_t vectors)

Example: Yearly inflation data looks a lot like a y_t

Our problem: estimation Given output y_t for $t = 1, 2, \ldots, T$, find the A_i's

Scalar models popular for macroeconomics; multivariate ones less used because of estimation difficulties

Transfer function formalism

To $A_0 u_t + A_1 u_{t-1} + A_2 u_{t-2} + \cdots + A_q u_{t-q}$ we associate

$$G(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_q\lambda^q$$

$G(\lambda)$ assumed nonsingular ($\det(A_0) \neq 0$), and all eigvals outside unit circle

Definition

$$G^\star(\lambda) := A_0^\top + A_1^\top\lambda^{-1} + A_2^\top\lambda^{-2} + \cdots + A_q^\top\lambda^{-q}$$

λ "behaves like" a complex number on the unit circle

$$G(\lambda)G^\star(\lambda) = M_{-q}\lambda^{-q} + \cdots + M_{-1}\lambda^{-1} + M_0 + M_1\lambda + \cdots + M_q\lambda^q$$

is palindromic ($M_0 = M_0^\top$ and $M_{-i} = M_i^\top$ for each i)

Autocovariance generating function

$$G(\lambda)G^\star(\lambda) = M_{-q}\lambda^{-q} + \cdots + M_{-1}\lambda^{-1} + M_0 + M_1\lambda + \cdots + M_q\lambda^q$$

Theorem [Box, Jenkins, '76 book]

$M_i = \mathbb{E}\left[y_t y_{t-i}^\top\right]$ for each $i = -q, \ldots, q$

1. Estimate $\tilde{M}_i := \frac{1}{T-1}\sum_{i=1}^{T} y_t y_{t-i}^\top$
2. Factorize $\tilde{M}(\lambda) = \tilde{G}(\lambda)\tilde{G}^\star(\lambda)$

Point 2. is spectral factorization [Wiener, Kailath, Kucera, Ran et al, Ephremidze et al; many more for the scalar case]

Spectral factorization and matrix equations

$$M_{-1}\lambda^{-1} + M_0 + M_1\lambda = (A_1\lambda + A_0)(A_1\lambda + A_0)^\star$$

Several approaches to solve it:

- $Y = A_0 A_0^\top$ solves $M_1 Y^{-1} M_1^\top + Y = M_0$ [Ran et al., Meini, CH Guo...]
- $Z = -A_0^{-\top} A_1^\top$ solves $M_{-1} + M_0 Z + M_1 Z^2$ [Bini et al, Higham and Kim, Meerbergen and Tisseur...]
- $G(\lambda)$ left spectral divisor of $M(\lambda)$ [Gohberg–Lancaster–Rodman]: linearization (palindromic?)+ Schur form

Extreme accuracy not necessary here — larger error from approximations $\tilde{M}_i$ already

What can go wrong

2. Factorize $\tilde{M}(\lambda) = \tilde{G}(\lambda)\tilde{G}^{\star}(\lambda)$

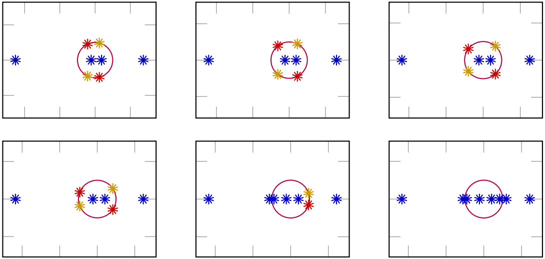
Factorization $\tilde{M}(\lambda) = \tilde{G}(\lambda)\tilde{G}^{\star}(\lambda)$ exists iff $\tilde{M}(z) \geq 0$ for each $z \in \mathbb{C}, |z| = 1$
$\tilde{M}_i$ are only approximate, so this property may fail

Possible solutions...

1. Apply a correction to $\tilde{M}(z)$
2. Robust factorization algorithm $\rightarrow$ do something in case of error (something like a "robust `chol`")
3. Get better $\tilde{M}_i$'s

Possible solutions

Apply a correction to $\tilde{M}(z)$ [Brüll, P, Sbrana, Schröder, preprint] regularization approach: perturb polynomial to move eigenvalues



Other approaches for similar structures around

[Alam et al, Benner and Voigt, Grivet-Talocia, Guglielmi-Overton. . .]

Robust factorization algorithm Not much usable around! [Kucera, 91]

Trying to modify some (Riccati-type eigensolvers, symbolic Cholesky factorization [Janashia, Lagvilava, Ephremidze '11])

Get better $\tilde{M}_i$'s

Other (slower) estimation methods typically get better $\tilde{M}_i$'s, such as Maximum Likelihood (ML)

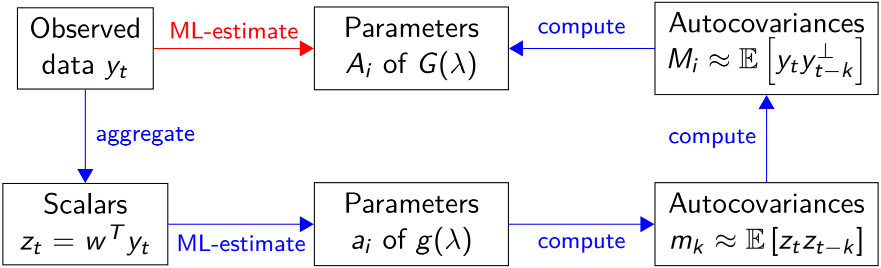
Our idea: get the $\tilde{M}_i$'s from projection + scalar ML

Algorithm

1. Choose weight row vector w, build scalar process $z_t = \{wy_t\}_{t=1,\dots,T}$
2. Use scalar ML to get $g(\lambda)$
3. $g(\lambda)g^\star(\lambda) = m(\lambda) = wM(\lambda)w^\top$
4. Repeat with many different w's to reconstruct $M(\lambda)$

Convergence properties to correct solution when $T \to \infty$ (technical) [P, Sbrana, submitted]

The algorithm in a picture



We take the blue road rather than the red

Many (easier) scalar problems rather than a (difficult) multivariate one

A 2×2 example

- $w = \begin{bmatrix} 1 & 0 \end{bmatrix}$:
$$wM(\lambda)w^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \lambda^{-1} + \begin{bmatrix} e & f \\ f & g \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} \lambda \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= a\lambda^{-1} + e + a\lambda$$

- $w = \begin{bmatrix} 0 & 1 \end{bmatrix}$:
$$wM(\lambda)w^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \lambda^{-1} + \begin{bmatrix} e & f \\ f & g \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} \lambda \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= d\lambda^{-1} + g + d\lambda$$

- $w = \begin{bmatrix} 1 & 1 \end{bmatrix}$:
$$m(\lambda) = (a+b+c+d)\lambda^{-1} + (e+2f+g) + (a+b+c+d)\lambda$$

If $b = c$, enough to get the remaining entries

Symmetric case

For a symmetric MA(1),

- $w = e_i$ for all i and $w = e_j + e_k$ for all $j \neq k$ enough to get $M(\lambda)$
- $A\lambda^{-1} + B + A\lambda$, with $A = A^\top, B = B^\top$ can be simultaneously diagonalized (congruence) $\rightarrow$ easier to find spectral factorization

This is enough to solve some easier models:

- Exponentially weighted moving average (random-walk-plus-noise) – MA(1) with symmetric matrices
- "Multiple trends" — MA(2) with symmetric matrices and $M_1 = -4M_2$

Nonsymmetric case

Idea Use $w(\lambda)$, not constant w!

$$\begin{bmatrix} 1 & \lambda \end{bmatrix} \begin{bmatrix} y_{t,1} \\ y_{t,2} \end{bmatrix} = y_{t,1} + y_{t-1,2}$$

- $w = \begin{bmatrix} 1 & \lambda \end{bmatrix}$

$$\begin{aligned} w(\lambda)M(\lambda)w^\star &= \begin{bmatrix} 1 & \lambda \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \lambda^{-1} + \begin{bmatrix} e & f \\ f & g \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} \lambda \right) \begin{bmatrix} 1 \\ \lambda^{-1} \end{bmatrix} \\ &= c\lambda^{-2} + (a+d+f)\lambda^{-1} + e + 2b + g \\ &\quad + (a+d+f)\lambda + c\lambda^2 \end{aligned}$$

$MA(q+1)$: more difficult to estimate, but still scalar

Least squares and more weights

Each aggregation vector $w(\lambda)$ gives us several equations

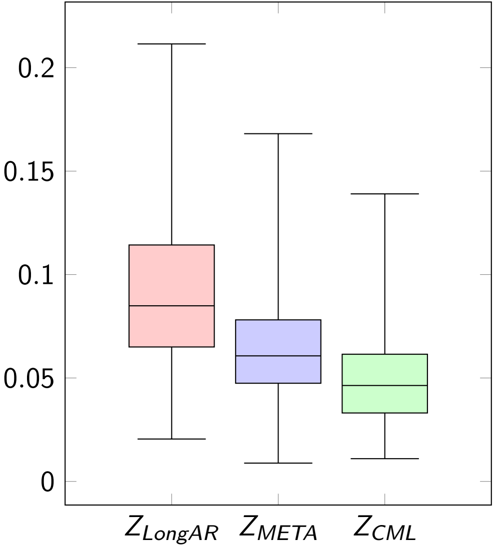
Combine them using least squares!

But... joint covariance?

$$W = \begin{bmatrix} \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \end{bmatrix}$$

Same-aggregation values (in red) easy to compute (independence)
Black ones are hard $\rightarrow$ set them to zero

Some results



LongAR Quicker but less accurate method

META Our method (Moment Estimation Through Aggregation)

CML Conditional Maximum Likelihood. Scales badly, basically impossible to beat

Frobenius norm relative errors, 300 trials, 2×2 problem, excluding 48 non-passive samples.

`eigvals(theta)=[-0.927492, -0.172508]`

Some more results

Symmetric model (EWMA), against out-of-the-box ML on the MA(1) representation (larger parameter space, difficult to enforce symmetry)

	T	Θ META	Θ ML	Σ_u META	Σ_u ML	Time META	Time ML
$d = 2$, easy	200	**202.52**	236.77	**108.28**	109.11	1.55	59.99
	400	**121.41**	138.13	83.31	**82.93**	3.13	107.28
	1000	**80.83**	101.53	**48.65**	48.96	8.34	226.64
$d = 2$, harder	200	**69.51**	78.26	**97.50**	98.31	1.36	46.53
	400	**48.26**	56.48	**80.91**	81.52	3.74	113.36
	1000	**28.01**	34.07	**47.60**	48.02	7.63	210.06
$d = 3$, easy	200	**205.07**	254.49	**135.26**	136.41	2.65	201.41
	400	**162.95**	187.40	93.48	**93.21**	5.77	316.74
	1000	**93.85**	108.92	**60.08**	61.13	12.62	523.67
$d = 3$, harder	200	**86.66**	107.22	**123.86**	124.19	2.81	202.95
	400	**57.03**	67.25	**95.13**	96.75	5.66	295.93
	1000	**29.91**	37.04	**61.78**	62.13	13.12	541.36

Frobenius norm parameter errors $\times 10^3$, average on 500 trials

+ real-life inflation test by Bankitalia

Parallelizability

```
@parallel for i = 1:nWeights
  # Estimate g_i(λ) with weight w_i
end
# Put together estimates to form M(λ) (fast)
# Spectral factorization (fast)
```

vs

```
while (convergence)
  #compute likelihood(G(λ))
  #compute G_{i+1}(λ) from G_i(λ) using BFGS-like search
end
function likelihood(G)
  for t  = 1:T
    # update state[t] from state[t-1], compute likelihood
  end
end
```

Why isn't this...

Why isn't this model reduction/system identification? Similar to tangential interpolation [Antoulas, Beattie, Gugercin '10 inbook] but...

- No access to input u_t
- No possibility to get $G(\lambda)$ exactly: $G(\lambda)Q$ equally good
- We work on a "squared" version: not clear how to relate $g(\lambda)$ and $G(\lambda)$ in our setting

Why isn't this compressed sensing? Similar to phase retrieval [Candès, Strohmer, Voroninski '11], but

- it is a "matrix version", uncertainty by an orthogonal matrix
- no sparsity/rank constraint, we want full reconstruction with a full set of measurements

Why isn't this Kalman filtering?

- KF assumes model known, our goal is estimating it here

Conclusions and issues

- Interesting idea to explore, new application area
- Works well in some cases (including symmetric problems)
- Still missing ideas: fit $G(\lambda)$ directly without squaring, find optimal weights and equations in LS
- Tricky to develop with the current software ecosystem
- Open problem: better “regularizing” spectral factorization algorithm:

  We know well how to solve Riccati-like equations, now we want to solve the ones that have no solution!

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Thanks for your attention