Perturbing Palindromic Matrix Equations to Make Them Solvable

Federico Poloni[1]
Joint work with Tobias Brüll[2], Giacomo Sbrana[3], Christian Schröder[4]

[1]U Pisa, Dept of Computer Science
[2]Computer Simulation Technology AG/ TU Berlin alumnus
[3]Rouen Business School, France
[4]TU Berlin, Germany

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VARMA(1,1) models

VARMA(1,1) [Lütkepohl, book '05]

$$x_t - \Phi x_{t-1} = u_t - \Theta u_{t-1}$$

x_t = observed variable $\in \mathbb{R}^d$
u_t = white noise (enough to assume uncorrelated) $\in \mathbb{R}^d$
$\Phi, \Theta \in \mathbb{R}^{d \times d}$, $\rho(\Phi) < 1$, $\rho(\Theta) < 1$

Many known models to simulate volatility reduce to VARMA(1,1):

- GARCH(1,1)
- Multivariate stochastic volatility models

Estimating VARMAs

Problem

Given enough observations (x_t) generated by a VARMA, determine parameters Φ, Θ

A common choice is QML (quasi-maximum-likelihood):

1. Assume u_t Gaussian independent
2. Given guesses $\hat{\Phi}, \hat{\Theta}$, compute likelihood $\ell(\hat{\Phi}, \hat{\Theta})$ of generating the given time series
3. Feed $\ell(\cdot, \cdot)$ into a black-box minimization procedure (e.g., Matlab's `fminunc`)

Problems with QML

1. Costly: each function evaluation costs $O(nd^3)$, with $n =$ length of time series. Hundreds or thousands required
2. Black-box: difficult to implement and tweak, and understand what's going on.
3. No convergence guarantees, non-convex optimization problem in many variables
4. Hey doc, what if our u_t isn't Gaussian independent?

Our attempt

Moment estimator: determine Φ, Θ as a function of the autocovariances

$$M_k = \mathbb{E}\left[x_t x_{t+k}^T\right]$$

We will show $(\Phi, \Theta) = f(M_0, M_1, M_2)$

GMM estimator

1. Compute sample autocovariances $\hat{M}_k = \frac{1}{n} \sum x_t x_{t+k}^T$
2. Get $(\hat{\Phi}, \hat{\Theta}) = f(\hat{M}_0, \hat{M}_1, \hat{M}_2)$

- Very fast: working only with $d \times d$ matrices, no dependence on n (after computing moments)
- Good asymptotic properties
- In simulated experiments, not as accurate as QML, but good as initial value / low complexity estimate

Already known for univariate GARCH; generalization requires some linear algebra machinery

Yule-Walker results

The parameter Φ is easy to obtain:

Theorem

$\Phi = M_{k+1} M_k^{-1}$ for each $k \geq 1$

Can solve any of these equations, e.g. $\hat{\Phi} = \hat{M}_2 \hat{M}_1^{-1}$
or many of them in the least-squares sense

If you heard about Hankel matrices and time series, that's where they arise

Estimating Θ

Let $r_t = x_t - \Phi x_{t-1} = u_t - \Theta u_{t-1}$, $Y := \mathbb{E}\left[u_t u_t^T\right]$

$$A_0 := \mathbb{E}_t\left[r_t r_t^T\right] = M_0 - \Phi M_1^T - M_1 \Phi^T + \Phi M_0 \Phi^T = Y + \Theta Y \Theta^T$$

$$A_1 := \mathbb{E}_t\left[r_t r_{t+1}^T\right] = M_1 - \Phi M_0 = -\Theta Y$$

Blue expressions allow us to compute A_0, A_1.
Use them + red expressions to decouple equations for Y, $X = \Theta^T$

$$A_0 = Y + A_1 Y^{-1} A_1^T, \quad Y > 0 \qquad \text{(BARE)}$$

$$A_1^T + A_0 X + A_1 X^2 = 0 \qquad \text{(UME)}$$

Two related matrix equations

$$A_0 = Y + A_1 Y^{-1} A_1^T, \quad Y > 0 \tag{BARE}$$

$$A_1^T + A_0 X + A_1 X^2 = 0 \tag{UME}$$

Solve any one of them, then $A_1 = -X^T Y$

(UME) looks more appealing, relation with quadratic eigenproblems
However, (BARE) more natural: no "hidden symmetry constraints"
[Engwerda *et al*, '93], [Meini, '02], [Guo *et al*, '10, '11, '12]

Spectral factorization problem

$$z^{-1} A_1^T + A_0 + z A_1 = (I - zX^T) Y (I - z^{-1} X)$$

Eigenvalue s of $I - zX^T$ outside the unit circle, $I - z^{-1}X$ inside

Existence of the solution

Existence and unicity

- Solution exists if $Q(\lambda) := A_1^T \lambda^{-1} + A_0 + A_1 \lambda$ is such that $Q(\lambda) > 0$ for each λ on unit circle [Engwerda *et al*, '93]
- Solution unique if we ask $Y > 0$, $\rho(X) < 1$ (as was assumed)

Of course, if the model is well-posed, there must be a solution...
But observed data $\hat{A}_0$, $\hat{A}_1$ might give unsolvable equations

Rather than giving up, perturb them to make the model solvable

Similar techniques (for other problems) in [Brüll, Schröder '12], [Alam, Bora, Byers, Overton '11]

Spectral plot

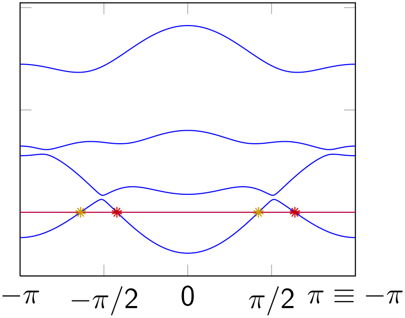


Figure: Eigenvalues of $Q(e^{i\omega})$

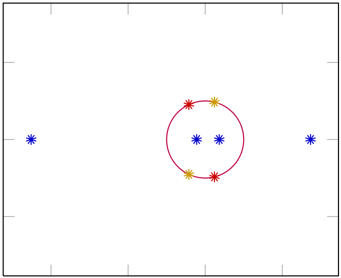
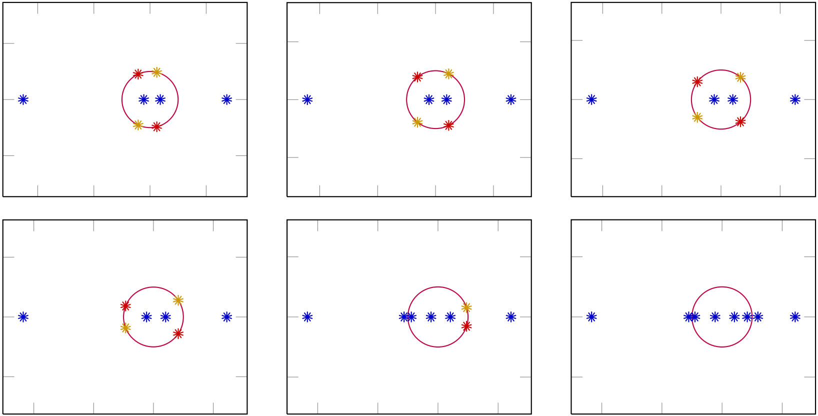


Figure: Generalized eigenvalues of $Q()$

Red/Yellow: sign characteristic of unimodular eigenvalues
Same thing as upward/downward slope in the graph on the left

Perturbing eigenvalues

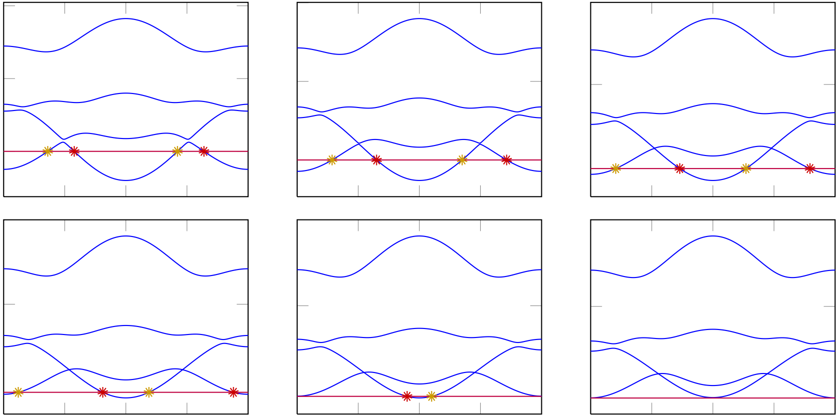
Perturbation behaviour: eigenvalues on the unit circle coalesce in pair to leave it



Plan: Perturb the matrices to make the eigenvalues coalesce — but how to pair them?

The other setting

Everything clearer if we look at the other plot



- Coalesce one red and one yellow point
- Red points move towards right, yellow ones towards left

Moving eigenvalues

Can use eigenvalue perturbation theory to predict (first-order) location of the unimodular eigenvalues after a perturbation

Theorem

If (λ, u) is a simple unimodular eigenpair of $\lambda^{-1}A_1^* + A_0 + \lambda A_1$, an eigenvalue of $\lambda^{-1}(A_1^* + E_1^*) + (A_0 + E_0) + \lambda(A_1 + E_1)$ is given by

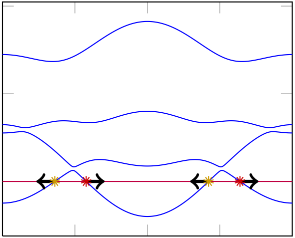
$$\tilde{\lambda} = \lambda - \frac{u^*(\lambda^{-1}E_1^* + E_0 + \lambda E_1)u}{u^*(-\lambda^{-2}A_1^* + A_1)u} + O(\|E_0, E_1\|)$$

Given a perturbation ansatz

$$A_i = \sum_k \delta_k E_i^{(k)}, \quad i = 0, 1$$

one can choose the δ_k such that the perturbed eigenvalues are (approximately) in a specified location (linear least-squares problem)

Iterative perturbation



$$A_i = \sum_k \delta_k E_i^{(k)}, \quad i = 0, 1$$

1. Choose step-size τ
2. Compute unimodular eigenvalues
3. Choose new desired location at distance τ in the right direction
4. Compute first-order location under each $(E_0^{(k)}, E_1^{(k)})$
5. Solve least-squares problem to compute δ_k that obtain best match
6. Repeat

A_i vs. M_i

Problem

$$A_0 := M_0 - \Phi M_1^T - M_1 \Phi^T + \Phi M_0 \Phi M^T$$
$$A_1 := M_1 - \Phi M_0 \qquad \Phi = M_2 M_1^{-1}$$

Perturbing A_i "unnatural", since they come from observed M_i

Solution work on extended equation

$$\lambda^{-1} \begin{bmatrix} M_1^T & M_0^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} M_0 & M_1 & M_2 \\ M_1^T & M_0 & M_1 \\ M_2^T & M_1^T & 0 \end{bmatrix} + \lambda \begin{bmatrix} M_1 & 0 & 0 \\ M_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Construct a linear perturbation basis $(E_0^{(k)}, E_1^{(k)})$ corresponding to entrywise perturbations of the M_i

Examples: closed-form estimator

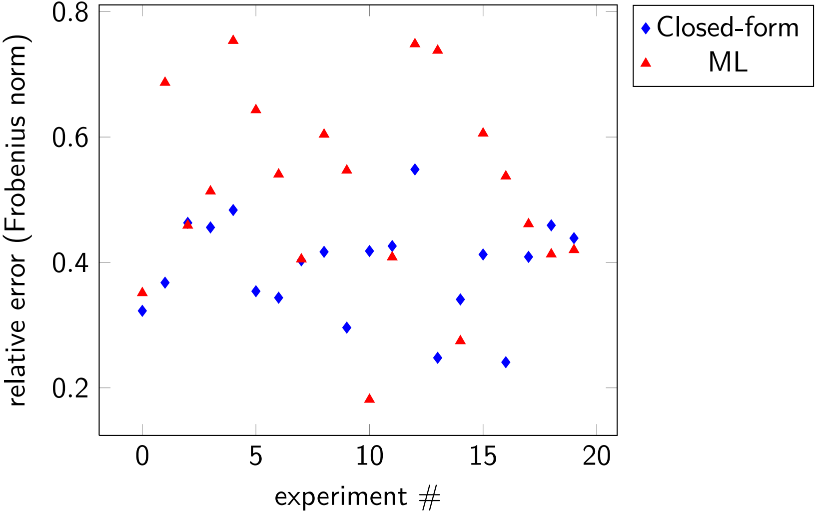


Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.6$, $n = 1000$

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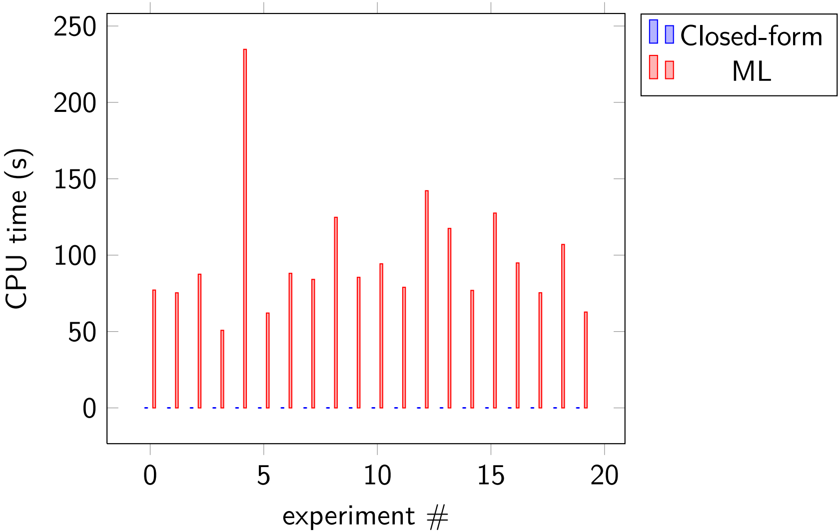

Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.6$, $n = 1000$

Examples: solvability enforcement

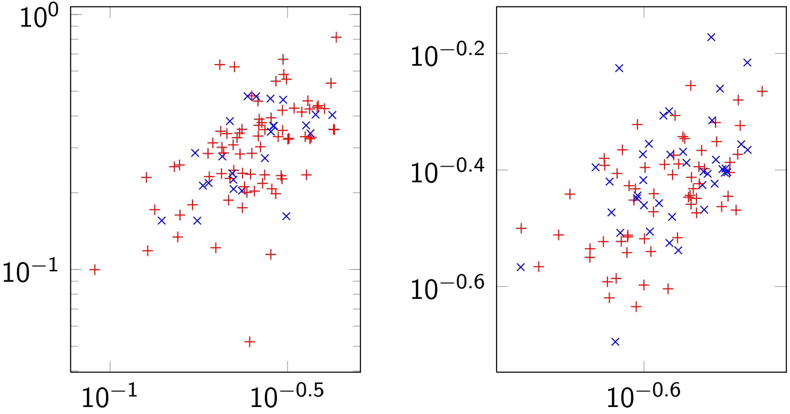

Figure: VARMA with $\rho(\Phi) = 0.9$, $\rho(\Theta) = 0.87$, $d = 2$ (left) or 4 (right), $n = 10000$. Blue x = enforcement needed

Examples: solvability enforcement

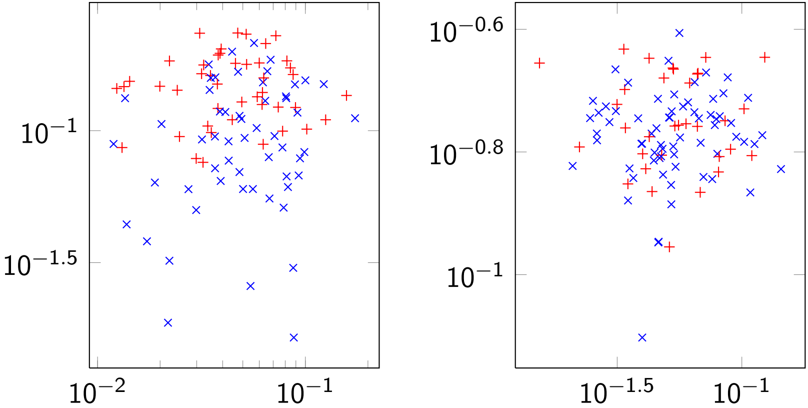

Figure: VARMA with $\rho(\Phi) = 0.9$, $\rho(\Theta) = 0.995$, $d = 2$ (left) or 4 (right), $n = 10000$. Blue x = enforcement needed

Possible improvements

- Work on Θ and Φ at the same time
- Combine with an iterative ML-like optimization
  e.g., GLS (generalized least squares) for GARCH?
- Spectral factorization with higher polynomial degrees

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Thanks for your attention!