When is a system of Sylvester-like matrix equations well posed?

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The setup

Example

$$\begin{cases} AX_1B + CX_2^\star D = E \\ FX_1^\star G + HX_4 I = J \\ KX_3^\star L + MX_2^\star N = O \end{cases}$$

- Each equation is in the form

$$AX_i^\sigma B + CX_j^\tau D = E.$$

- $A, B, C, D, E \in \mathbb{C}^{n\times n}$ (n constant along all equations);
- Unknowns $X_1, X_2, \dots, X_m \in \mathbb{C}^{n\times n}$. Two of them in each equation.
- Symbols $\sigma, \tau \in \{1, \star\}$, where $\star$ is either $*$ (transpose conjugate) or $\top$ (complex transpose).

Where do they arise?

Stability of dynamical systems:

Theorem (Lyapunov stability)

Given any pos.def. Q, the linear continuous-time system $\dot{x} = Ax$ is asymptotically stable iff $A^*X + XA + Q = 0$ has a unique pos.def. solution.

Theorem (Stein stability)

Given any pos.def. Q, the linear discrete-time system $x_{k+1} = Ax_k$ is asymptotically stable iff $A^*XA - X + Q = 0$ has a unique pos.def. solution.

Where do they arise? — II

Decoupling systems of equations:

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \longrightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Is there a change of basis that does this?

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AX - XB + C \\ 0 & B \end{bmatrix}.$$

(Consider 1×1 case for first solvability conditions)

Where do they arise? — III

Decoupling systems of equations:

$$\begin{bmatrix} E & G \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \longrightarrow \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Is there a change of basis that does this?

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} E & G \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} E & EX - YF + G \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} A & AX - YB + C \\ 0 & B \end{bmatrix}.$$

Where do they arise? — IV

Decoupling systems of equations of the form $M^{-1}M^{\star}$:

$$\begin{bmatrix} 0 & A \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^{\star} \\ A^{\star} & C^{\star} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^{\star} \\ A^{\star} & 0 \end{bmatrix}.$$

Is there a change of basis that does this?

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & A \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ X^{\star} & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ X^{\star} & I \end{bmatrix} \begin{bmatrix} 0 & B^{\star} \\ A^{\star} & C^{\star} \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A \\ B & X^{\star}A + BX + C \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^{\star} \\ A^{\star} & (X^{\star}A + BX + C)^{\star} \end{bmatrix}.$$

Vectorization

If $\star = \top$, it's a linear system: vectorize.

Example

$$\begin{cases} AX_1B + CX_2^\top D = E \\ FX_2^\top G + HX_1^\top I = J \end{cases} \iff \begin{bmatrix} B^\top \otimes A & (D^\top \otimes C)\Pi \\ (G^\top \otimes F)\Pi & (I^\top \otimes H)\Pi \end{bmatrix} \begin{bmatrix} \operatorname{vec} X_1 \\ \operatorname{vec} X_2 \end{bmatrix} = \begin{bmatrix} \operatorname{vec} E \\ \operatorname{vec} J \end{bmatrix}$$

Otherwise, $\mathbb{R}$-linear: divide real and imaginary parts and vectorize.

Example

$$A\overline{X}B = E \iff \begin{bmatrix} \Re B^\top \otimes \Re A - \Im B^\top \otimes \Im A & \Im B^\top \otimes \Re A + \Re B^\top \otimes \Im A \\ \Im B^\top \otimes \Re A + \Re B^\top \otimes \Im A & \Im B^\top \otimes \Im A - \Re B^\top \otimes \Re A \end{bmatrix} \begin{bmatrix} \operatorname{vec} \Re X \\ \operatorname{vec} \Im X \end{bmatrix} = \begin{bmatrix} \operatorname{vec} \Re E \\ \operatorname{vec} \Im E \end{bmatrix}$$

Well-posedness

System of generalized $\star$-Sylvester equations $\iff$ huge $Mx = v$.

Main question

When is M square invertible? i.e.,

When is the system uniquely solvable for each RHS?

And how can one compute the solution?

Meaningful case for numerical practice, because it means the answer is stable under small perturbations.

First constraint: number of equations = number of unknowns.

Roth-style conditions

Easy answer: the condition is $\det M \neq 0$.
Not satisfying: M is at least $rn^2 \times rn^2$.
We want conditions based on matrices and pencils of size $O(rn)$. For instance:

Theorem [Classical]

$AX - XD = E$ well-posed iff A and D have no common eigenvalues.

Theorem [De Terán, Iannazzo, LAA '16]

$AXB + CX^\star D = E$ well-posed iff $Q(\lambda) = \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$ has no pair of eigenvalues λ_1, λ_2 such that $\lambda_1 \lambda_2^\star = 1$.

Exception: if $\star = \top$, one copy of 1 and -1 allowed.

One equation: the Bartels–Stewart algorithm

[Bartels–Stewart '72]

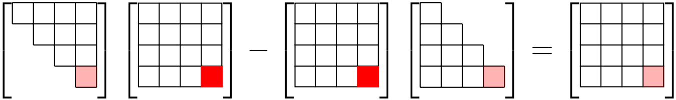
Step 1: reduce to triangular coefficients

Theorem [Schur]

Let $A \in \mathbb{C}^{n\times n}$. There is a unitary Q such that $\hat{A} = Q^* AQ$ is upper (lower) triangular.

$$AX - XD = E \longrightarrow Q_A^* AXQ_D - Q_A^* DQ_D = Q_A^* EQ_D$$
$$\longrightarrow \hat{A}\hat{X} - \hat{X}\hat{D} = \hat{E}, \quad \hat{X} = Q_A^* XQ_D,\ \hat{E} = Q_A^* EQ_D.$$

Step 2: back-substitution



$$\hat{A}_{44}\hat{X}_{44} - \hat{X}_{44}\hat{D}_{44} = \hat{E}_{44}$$

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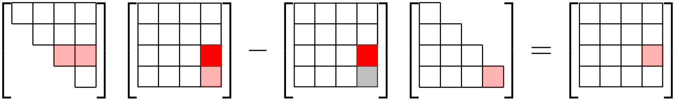

$$\hat{A}_{33}\hat{X}_{34} - \hat{X}_{34}\hat{D}_{44} = \hat{E}_{34} - \hat{A}_{34}\hat{X}_{44}$$

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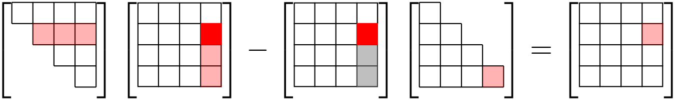

$$\hat{A}_{ii}\hat{X}_{ij} - \hat{X}_{ij}\hat{D}_{jj} = \hat{E}_{ij} - \dots$$

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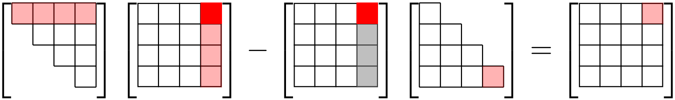

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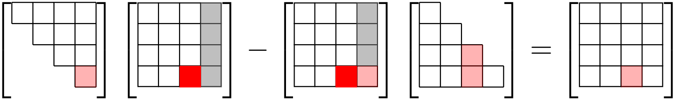

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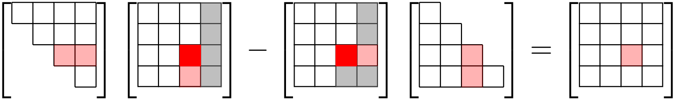

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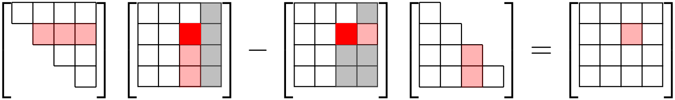

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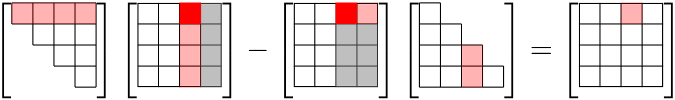

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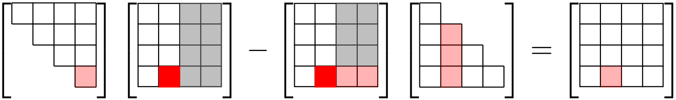

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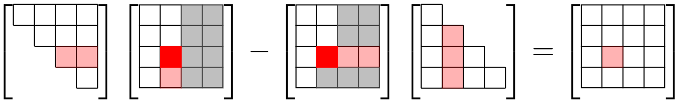

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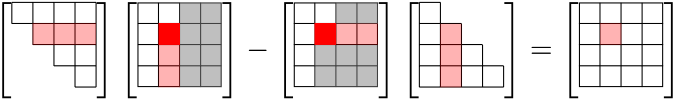

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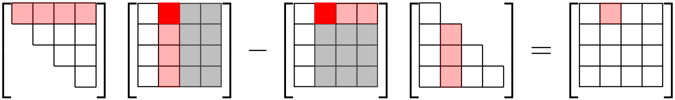

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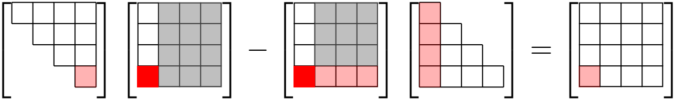

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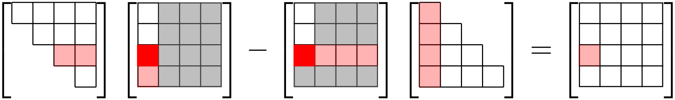

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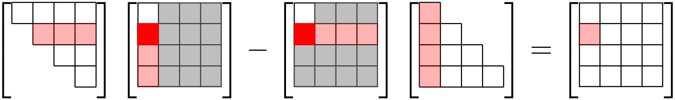

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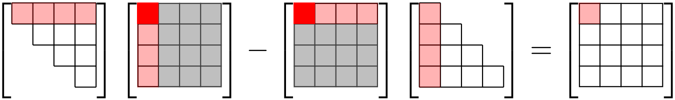

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Solvability conditions

Equation for X_{ij} uniquely solvable iff $\hat{A}_{ii} - \hat{D}_{jj} \neq 0$

Theorem [Bartels–Stewart '72]

$AX - XD = C$ well-posed iff A and D have no common eigenvalues.

... and a $\mathcal{O}(n^3)$ algorithm to solve it.

Now, back to systems:

Example

$$\begin{cases} AX_1B + CX_2^\star D = E \\ FX_1^\star G + HX_4I = J \\ KX_3^\star L + MX_2^\star N = O \end{cases}$$

Eliminate single-use variables

Suppose X_1 appears only in one equation

$$AX_1^\sigma B + CX_j^\tau D = E.$$

A singular: Not well posed: if $Au = 0$, we can add multiples of $uu^\star$ to X_1.

B singular: Not well posed, similarly.

A, B invertible: determine uniqueness for the remaining equations, then solve for X_1.

So we can assume each variable appears ~~at least~~ exactly two times.

Reduce to cycles

Example

$$\begin{cases} AX_1^\star B + CX_4 D = 0 \\ EX_2^\star F + GX_3 H = 0 \\ IX_5 J + KX_4^\star L = 0 \\ MX_1 N + OX_5^\star P = 0 \\ QX_3^\star R + SX_2 T = 0 \end{cases} \qquad \begin{cases} \\ \\ \\ \\ \\ \end{cases}$$

$$Y_1 = X_1,$$

Reduce to cycles

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$$Y_1 = X_1, \; Y_2 = X_4^\star,$$

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$$\begin{cases} AX_1^\star B + CX_4 D = 0 \\ EX_2^\star F + GX_3 H = 0 \\ IX_5 J + KX_4^\star L = 0 \\ MX_1 N + OX_5^\star P = 0 \\ QX_3^\star R + SX_2 T = 0 \end{cases} \qquad \begin{cases} B^\star Y_1 A^\star + D^\star Y_2 C^\star = 0 \\ KY_2 L + IX_5 J = 0 \\ \\ \\ \\ \end{cases}$$

$$Y_1 = X_1,\ Y_2 = X_4^\star,$$

Reduce to cycles

Example

$$\begin{cases} AX_1^\star B + CX_4 D = 0 \\ EX_2^\star F + GX_3 H = 0 \\ IX_5 J + KX_4^\star L = 0 \\ MX_1 N + OX_5^\star P = 0 \\ QX_3^\star R + SX_2 T = 0 \end{cases} \qquad \begin{cases} B^\star Y_1 A^\star + D^\star Y_2 C^\star = 0 \\ KY_2 L + I{\color{red}Y_3} J = 0 \\ \\ \\ \\ \end{cases}$$

$$Y_1 = X_1,\ Y_2 = X_4^\star,\ {\color{red}Y_3 = X_5}$$

Reduce to cycles

Example

$$\begin{cases} AX_1^\star B + CX_4 D = 0 \\ EX_2^\star F + GX_3 H = 0 \\ IX_5 J + KX_4^\star L = 0 \\ MX_1 N + OX_5^\star P = 0 \\ QX_3^\star R + SX_2 T = 0 \end{cases} \qquad \begin{cases} B^\star Y_1 A^\star + D^\star Y_2 C^\star = 0 \\ KY_2 L + IY_3 J = 0 \end{cases}$$

$$Y_1 = X_1,\ Y_2 = X_4^\star,\ Y_3 = X_5$$

Reduce to cycles

Example

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Reduce to cycles

Example

$$\begin{cases} AX_1^\star B + CX_4 D = 0 \\ EX_2^\star F + GX_3 H = 0 \\ IX_5 J + KX_4^\star L = 0 \\ MX_1 N + OX_5^\star P = 0 \\ QX_3^\star R + SX_2 T = 0 \end{cases}$$

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$$\begin{cases} B^\star Y_1 A^\star + D^\star Y_2 C^\star = 0 \\ KY_2 L + IY_3 J = 0 \\ P^* Y_3 O^* + N^* Y_1^\star M^* = 0 \\ \hline EX_2^\star F + GX_3 H = 0 \\ QX_3^\star R + SX_2 T = 0 \end{cases}$$

$$Y_1 = X_1,\ Y_2 = X_4^\star,\ Y_3 = X_5$$

Reduction

Theorem

A Sylvester-like system can be reduced to several disjoint 'cyclic' systems each with at most one nontrivial $\sigma \in \{1, \star\}$.

$$
\begin{cases}
A_1 Y_1 B_1 - C_1 Y_2 D_1 = 0, \\
A_2 Y_2 B_2 - C_2 Y_3 D_2 = 0, \\
\qquad \vdots \\
A_{r-1} Y_{r-1} B_{r-1} - C_{r-1} Y_r D_{r-1} = 0, \\
A_r Y_r B_r - C_r Y_1^{\sigma} D_r = 0.
\end{cases}
$$

$\sigma = 1$: periodic Sylvester equations

If $\sigma = 1$, solved in [Byers-Rhee '95].
Idea: generalized Bartels–Stewart algorithm.

Step 1: triangulation
Make A_i, C_i upper triangular and B_i, D_i lower triangular.

Theorem (Periodic Schur decomposition, [Bojanczyk-Golub-Van Dooren '92])

Let $A_1, A_2, \dots, A_r, C_1, C_2, \dots, C_r \in \mathbb{C}^{n\times n}$. There exist unitary matrices $Q_1, Q_2, \dots, Q_r, Z_1, Z_2, \dots, Z_r \in \mathbb{C}^{n\times n}$ such that

$$Q_1 A_1 Z_1, Q_2 A_2 Z_2, \dots, Q_r A_r Z_r,$$
$$Q_1 C_1 Z_2, Q_2 C_2 Z_3, \dots, Q_r C_r Z_1$$

are all upper (or: lower) triangular.

The change of bases

Or: insert orthogonal transformations in $P = C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1$ to make each factor upper triangular:

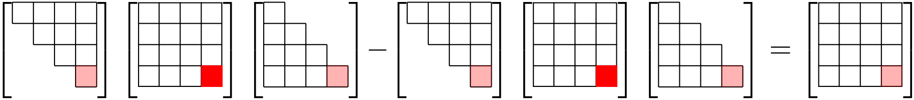
$$\underbrace{Z_1^* C_r^{-1} Q_r^*}_{\hat{C}_r^{-1}} \underbrace{Q_r A_r Z_r}_{\hat{A}_r} \underbrace{Z_r^* C_{r-1}^{-1} Q_{r-1}^*}_{\hat{C}_{r-1}^{-1}} \underbrace{Q_{r-1} A_{r-1} Z_{r-1}}_{\hat{A}_{r-1}} \cdots \underbrace{Z_2^* C_1^{-1} Q_1^*}_{\hat{C}_1^{-1}} \underbrace{Q_1 A_1 Z_1}_{\hat{A}_1}.$$

The eigenvalues of P are $\lambda_i = \frac{(\hat{A}_r)_{ii}(\hat{A}_{r-1})_{ii}\cdots(\hat{A}_1)_{ii}}{(\hat{C}_r)_{ii}(\hat{C}_{r-1})_{ii}\cdots(\hat{C}_1)_{ii}}$, $i = 1, 2, \dots, n$.

We can do this formally even if some C_k are singular (λ_i may be ∞).

Omitted in this talk: $\lambda_i = \frac{0}{0}$, singular formal product.
(Spoiler: it's never well-posed.)

Step 2: the back-substitution

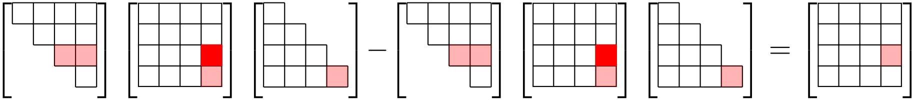


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{nn}(B_1)_{nn} & -(C_1)_{nn}(D_1)_{nn} & & \\ & (A_2)_{nn}(B_2)_{nn} & -(C_2)_{nn}(D_2)_{nn} & \\ & & \ddots & \ddots \\ -(C_r)_{nn}(D_r)_{nn} & & & (A_r)_{nn}(B_r)_{nn} \end{bmatrix} \begin{bmatrix} (Y_1)_{nn} \\ (Y_2)_{nn} \\ \vdots \\ (Y_r)_{nn} \end{bmatrix} = R.H.S.$$

Invertible if $(A_1)_{nn}(B_1)_{nn} \cdots (A_r)_{nn}(B_r)_{nn} \neq (C_1)_{nn}(D_1)_{nn} \cdots (C_r)_{nn}(D_r)_{nn}$

Step 2: the back-substitution

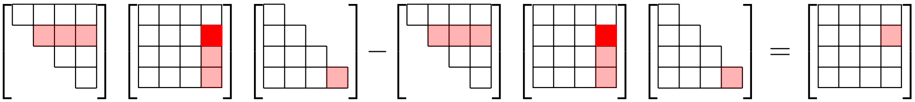


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

Step 2: the back-substitution

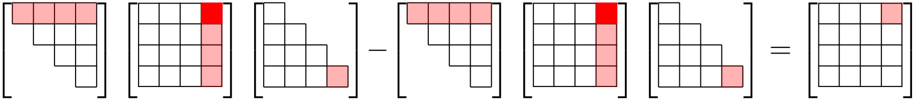


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

Step 2: the back-substitution



Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

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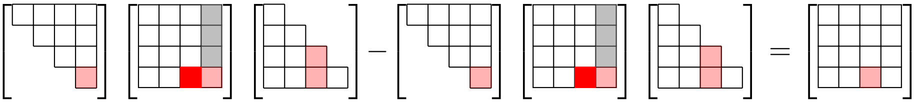

Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

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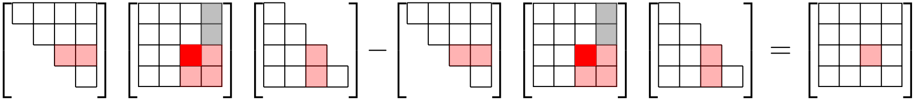

Cyclic bidiagonal system

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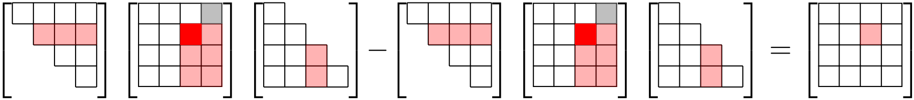
Invertible if $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

Step 2: the back-substitution



Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

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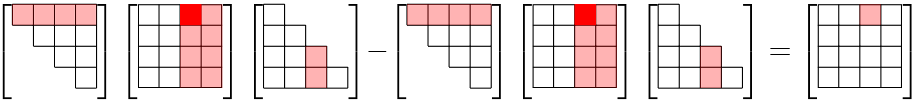

Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

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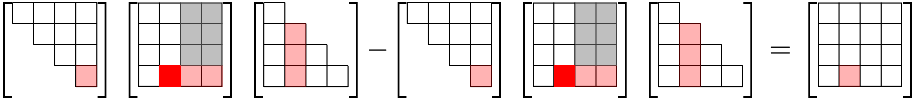

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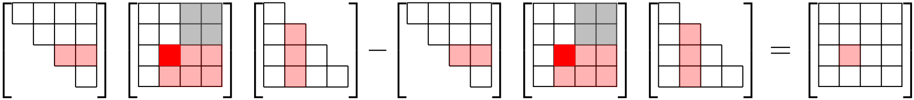
Step 2: the back-substitution



Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

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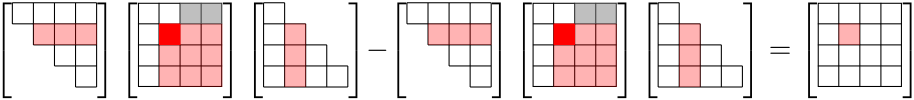

Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

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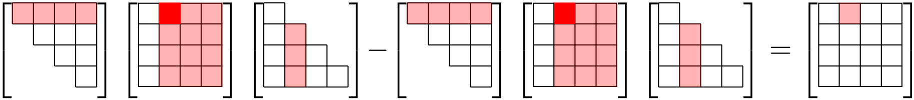

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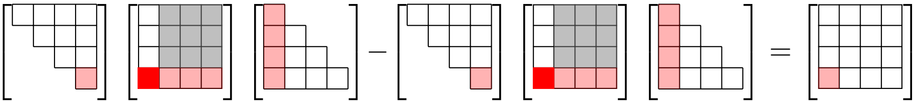

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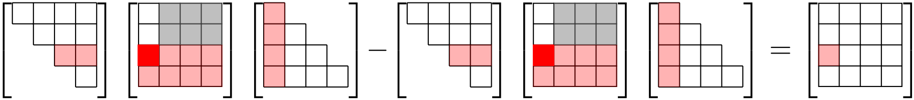
Step 2: the back-substitution



Cyclic bidiagonal system

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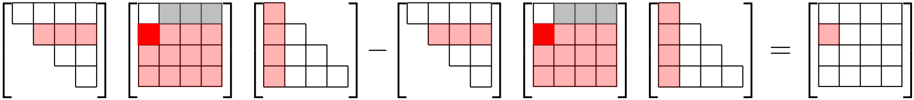

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$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & \\ & & \ddots & \ddots \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

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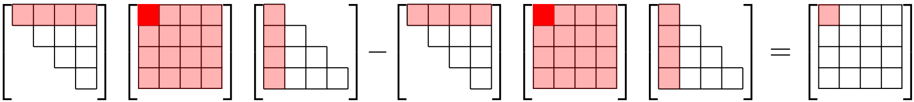

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Well-posedness of systems with $\sigma = 1$

Theorem [Byers-Rhee '95]

The cyclic system

$$\begin{cases} A_1 Y_1 B_1 - C_1 Y_2 D_1 = 0, \\ A_2 Y_2 B_2 - C_2 Y_3 D_2 = 0, \\ \qquad \vdots \\ A_{r-1} Y_{r-1} B_{r-1} - C_{r-1} Y_r D_{r-1} = 0, \\ A_r Y_r B_r - C_r Y_1 D_r = 0. \end{cases}$$

is well posed iff the formal products

$$R = C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1$$
$$S = D_r B_r^{-1} D_{r-1} B_{r-1}^{-1} \cdots D_1 B_1^{-1}$$

have no common eigenvalues.

$\sigma = \star$: periodic $\star$-Sylvester systems

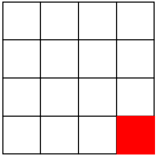
Step 1: compute periodic Schur form of

$$S^{-\star}R = D_r^{-\star}B_r^{\star}D_{r-1}^{-\star}B_{r-1}^{\star}\cdots D_1^{-\star}B_1^{\star}C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1.$$

Step 2: block back-substitution: solve a $2r \times 2r$ system for

$$(Y_1)_{ij}, (Y_2)_{ij}, \dots (Y_r)_{ij}, (Y_1)_{ji}, (Y_2)_{ji}, \dots, (Y_r)_{ji}$$

simultaneously.



$\sigma = \star$: periodic $\star$-Sylvester systems

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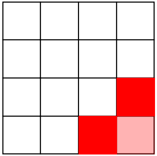

$\sigma = \star$: periodic $\star$-Sylvester systems

Step 1: compute periodic Schur form of

$$S^{-\star}R = D_r^{-\star}B_r^{\star}D_{r-1}^{-\star}B_{r-1}^{\star}\cdots D_1^{-\star}B_1^{\star}C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1.$$

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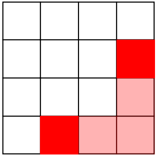

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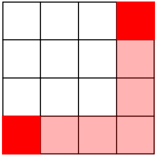

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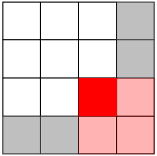

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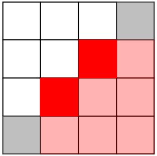

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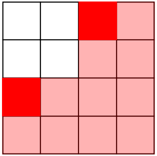

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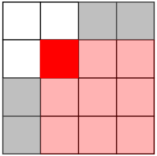

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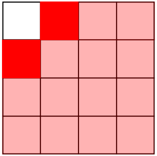

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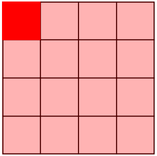

The well-posedness conditions

Reciprocal-free conditions:

Theorem

Let λ_i, $i = 1, 2, \ldots, n$ be the eigenvalues of the formal matrix product $S^{-\star}R$. A cyclic $\star$-Sylvester system is well-posed iff

$\boxed{\star = *}$: $\lambda_i \overline{\lambda_j} \neq 1 \; \forall\, i, j$:

$$\lambda_i \overline{\lambda_i} \neq 1 \qquad \forall\, i \qquad \textit{(from the diagonal blocks)}$$
$$\lambda_i \overline{\lambda_j} \neq 1 \qquad \forall\, i \neq j \qquad \textit{(from the off-diagonal blocks)}$$

$\boxed{\star = \top}$: $\lambda_i \lambda_j \neq 1 \; \forall\, i, j$, *but* $\lambda = -1$ *may appear once*:

$$\lambda_i \neq 1 \qquad \forall\, i \qquad \textit{(from the diagonal blocks)}$$
$$\lambda_i \lambda_j \neq 1 \qquad \forall\, i \neq j \qquad \textit{(from the off-diagonal blocks)}$$

Another approach: duplicate the system

Example

$$
\begin{aligned}
A_1 Y_1 B_1 - C_1 Y_2 D_1 &= E_1 \\
A_2 Y_2 B_2 - C_2 Y_3 D_2 &= E_2 \\
A_3 Y_3 B_3 - C_3 Y_1^* D_3 &= E_3
\end{aligned}
$$

Another approach: duplicate the system

Example

$$
\begin{aligned}
A_1 Y_1 B_1 - C_1 Y_2 D_1 &= E_1 \qquad & B_1^* Y_1^* A_1^* - D_1^* Y_2^* C_1^* &= E_1^* \\
A_2 Y_2 B_2 - C_2 Y_3 D_2 &= E_2 \qquad & B_2^* Y_2^* A_2^* - D_2^* Y_3^* C_2^* &= E_2^* \\
A_3 Y_3 B_3 - C_3 Y_1^* D_3 &= E_3 \qquad & B_3^* Y_3^* A_3^* - D_3^* Y_1 C_3^* &= E_3^*
\end{aligned}
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Another approach: duplicate the system

Example

$$
\begin{aligned}
A_1 Y_1 B_1 - C_1 Y_2 D_1 &= E_1 & \qquad B_1^* Z_1 A_1^* - D_1^* Z_2 C_1^* &= E_1^* \\
A_2 Y_2 B_2 - C_2 Y_3 D_2 &= E_2 & \qquad B_2^* Z_2 A_2^* - D_2^* Z_3 C_2^* &= E_2^* \\
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Another approach: duplicate the system

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$$\begin{aligned} A_1 Y_1 B_1 - C_1 Y_2 D_1 &= E_1 & \qquad B_1^* Z_1 A_1^* - D_1^* Z_2 C_1^* &= E_1^* \\ A_2 Y_2 B_2 - C_2 Y_3 D_2 &= E_2 & \qquad B_2^* Z_2 A_2^* - D_2^* Z_3 C_2^* &= E_2^* \\ A_3 Y_3 B_3 - C_3 Z_1 D_3 &= E_3 & \qquad B_3^* Z_3 A_3^* - D_3^* Y_1 C_3^* &= E_3^* \end{aligned}$$

Lemma (only for $\star = *$)

The 'duplicated' Sylvester system has a unique solution iff the original $*$-Sylvester system has unique solution.

Solvability condition for the duplicated system: $S^{-*}R$ and $R^{-*}S$ have no common eigenvalues.

Gives immediately the well-posedness condition $\lambda_i \overline{\lambda_j} \neq 1 \; \forall \, i, j$.

The duplication lemma

Lemma (only for $\star = *$)

The 'duplicated' Sylvester system has a unique solution iff the original $*$-Sylvester system has unique solution.

Nontrivial part: we need to prove that if

$$\begin{aligned} A_1 Y_1 B_1 - C_1 Y_2 D_1 &= 0 & B_1^* Z_1 A_1^* - D_1^* Z_2 C_1^* &= 0 \\ A_2 Y_2 B_2 - C_2 Y_3 D_2 &= 0 & B_2^* Z_2 A_2^* - D_2^* Z_3 C_2^* &= 0 \\ A_3 Y_3 B_3 - C_3 Z_1 D_3 &= 0 & B_3^* Z_3 A_3^* - D_3^* Y_1 C_3^* &= 0 \end{aligned}$$

has a nonzero solution, then it has one with property $(*)$: $Z_i = Y_i^* \ \forall\, i$.

$(Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*)$ is another solution.

By linearity, $(Y_1 + Z_1^*, Y_2 + Z_2^*, Y_3 + Z_3^*, Z_1 + Y_1^*, Z_2 + Y_2^*, Z_3 + Y_3^*)$ is a solution, and has property $(*)$.

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Let's try again: By linearity,
$(i(Y_1 - Z_1^*), i(Y_2 - Z_2^*), i(Y_3 - Z_3^*), i(Z_1 - Y_1^*), i(Z_2 - Y_2^*), i(Z_3 - Y_3^*))$ is a solution, and has property $(*)$.

Duplication lemma - counterexample

We needed the fact that $(aX)^* = \overline{a}X^*$. For $\top$, it doesn't work.

Counterexample

$$2y + 2y = 0$$ has unique solution,

$$\begin{cases} 2y + 2z = 0 \\ 2z + 2y = 0 \end{cases}$$ does not.

Recap

- Well-posedness of $\star$-Sylvester systems reduced to cyclic case.
- Depends only on eigenvalues of certain formal products.
- Bartels–Stewart-like algorithm (non-structurally backward stable).
- Duplication algorithm ($\star = *$) (non-structurally backward stable).
- Same ideas work for systems with mixed $\top$, $*$ and $^-$ symbols.
- More than two terms per equation: more involved already for $r = 1$.
- Non-square case: more involved already for $r = 1$, see [De Terán, Iannazzo, P, Robol, Arxiv '16].

The non-square case, $r = 1$

Theorem ([De Terán, Iannazzo, Poloni, Robol '16 Arxiv])

$AXB + CX^\star D = E$ well-posed iff the following conditions hold:

- *Either (A and B) or (C and D) are square*
- *The smaller of these two square coefficients is invertible*
- $\mathcal{Q}(\lambda) = \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}$ *is regular with 'reciprocal-free' eigenvalues.*

$\mathcal{Q}(\lambda)$ 'replaces' $D^{-*}B^*C^{-1}A$ (tricky to define: [Granat Kågström Kressner '07]).

Eigenvalues alone not sufficient to determine well-posedness:

Counterexample

$$\mathcal{Q}_1(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 0 \\ \hline 1 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right], \qquad \mathcal{Q}_2(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 1 \\ \hline 0 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right].$$

Why only two terms?

If we can solve systems with 3 terms, we can solve equations with arbitrarily many, by adding variables.

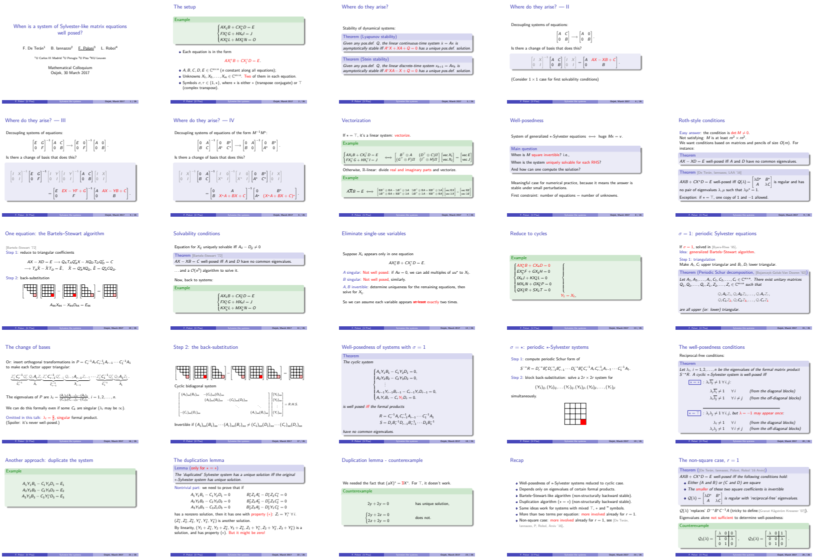
Example

$$AXB + CXD + EXF + GXH + IXJ + KXL = M$$

$$\downarrow$$

$$\begin{aligned} AXB + CXD + Y &= 0, \\ EXF + GXH + Z &= 0, \\ IXJ + KXL + W &= 0, \\ Y + Z + W &= M. \end{aligned}$$

Related to a big open problem (what is the complexity $\mathcal{O}(n^\tau)$ of matrix multiplication?)



Thanks for your attention!

Questions?