

# When is a system of Sylvester-like matrix equations well posed?

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# The setup

## Example

$$\begin{cases} AX_1B + CX_2^*D = E \\ FX_1^*G + HX_4I = J \\ KX_3^*L + MX_2^*N = O \end{cases}$$

- Each equation is in the form

$$AX_i^\sigma B + CX_j^\tau D = E.$$

- $A, B, C, D, E \in \mathbb{C}^{n \times n}$  ( $n$  constant along all equations);
- Unknowns  $X_1, X_2, \dots, X_m \in \mathbb{C}^{n \times n}$ . **Two** of them in each equation.
- Symbols  $\sigma, \tau \in \{1, \star\}$ , where  $\star$  is either  $*$  (transpose conjugate) or  $\top$  (complex transpose).

# Where do they arise?

Stability of dynamical systems:

## Theorem (Lyapunov stability)

Given any pos.def.  $Q$ , the linear continuous-time system  $\dot{x} = Ax$  is asymptotically stable iff  $A^*X + XA + Q = 0$  has a unique pos.def. solution.

## Theorem (Stein stability)

Given any pos.def.  $Q$ , the linear discrete-time system  $x_{k+1} = Ax_k$  is asymptotically stable iff  $A^*XA - X + Q = 0$  has a unique pos.def. solution.

## Where do they arise? — II

Decoupling systems of equations:

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \longrightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Is there a change of basis that does this?

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AX - XB + C \\ 0 & B \end{bmatrix}.$$

(Consider  $1 \times 1$  case for first solvability conditions)

## Where do they arise? — III

Decoupling systems of equations:

$$\begin{bmatrix} E & G \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \longrightarrow \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Is there a change of basis that does this?

$$\begin{aligned} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} E & G \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \\ = \begin{bmatrix} E & EX - YF + G \\ 0 & F \end{bmatrix}^{-1} \begin{bmatrix} A & AX - YB + C \\ 0 & B \end{bmatrix}. \end{aligned}$$

## Where do they arise? — IV

Decoupling systems of equations of the form  $M^{-1}M^*$ :

$$\begin{bmatrix} 0 & A \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^* \\ A^* & C^* \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^* \\ A^* & 0 \end{bmatrix}.$$

Is there a change of basis that does this?

$$\begin{aligned} & \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & A \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ X^* & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ X^* & I \end{bmatrix} \begin{bmatrix} 0 & B^* \\ A^* & C^* \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & A \\ B & X^*A + BX + C \end{bmatrix}^{-1} \begin{bmatrix} 0 & B^* \\ A^* & (X^*A + BX + C)^* \end{bmatrix}. \end{aligned}$$

# Vectorization

If  $\star = \top$ , it's a linear system: **vectorize**.

## Example

$$\begin{cases} AX_1B + CX_2^\top D = E \\ FX_2^\top G + HX_1^\top I = J \end{cases} \iff \begin{bmatrix} B^\top \otimes A & (D^\top \otimes C)II \\ (G^\top \otimes F)II & (I^\top \otimes H)II \end{bmatrix} \begin{bmatrix} \text{vec } X_1 \\ \text{vec } X_2 \end{bmatrix} = \begin{bmatrix} \text{vec } E \\ \text{vec } J \end{bmatrix}$$

Otherwise,  $\mathbb{R}$ -linear: divide **real and imaginary parts** and vectorize.

## Example

$$A\bar{X}B = E \iff \begin{bmatrix} \Re B^\top \otimes \Re A - \Im B^\top \otimes \Im A & \Im B^\top \otimes \Re A + \Re B^\top \otimes \Im A \\ \Im B^\top \otimes \Re A + \Re B^\top \otimes \Im A & \Im B^\top \otimes \Im A - \Re B^\top \otimes \Re A \end{bmatrix} \begin{bmatrix} \text{vec } \Re X \\ \text{vec } \Im X \end{bmatrix} = \begin{bmatrix} \text{vec } \Re E \\ \text{vec } \Im E \end{bmatrix}$$

# Well-posedness

System of generalized  $\star$ -Sylvester equations  $\iff$  huge  $Mx = v$ .

## Main question

When is  $M$  **square invertible**? i.e.,

When is the system **uniquely solvable for each RHS**?

And how can one compute the solution?

Meaningful case for numerical practice, because it means the answer is stable under small perturbations.

First constraint: number of equations = number of unknowns.



## Roth-style conditions

Easy answer: the condition is  $\det M \neq 0$ .

Not satisfying:  $M$  is at least  $rn^2 \times rn^2$ .

We want conditions based on matrices and pencils of size  $O(rn)$ . For instance:

**Theorem** [Classical]

$AX - XD = E$  well-posed iff  $A$  and  $D$  have no common eigenvalues.

**Theorem** [De Terán, Iannazzo, LAA '16]

$AXB + CX^*D = E$  well-posed iff  $Q(\lambda) = \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}$  has no pair of eigenvalues  $\lambda_1, \lambda_2$  such that  $\lambda_1 \lambda_2^* = 1$ .

Exception: if  $\star = \top$ , one copy of 1 and  $-1$  allowed.

# One equation: the Bartels–Stewart algorithm

[Bartels–Stewart '72]

Step 1: reduce to triangular coefficients

Theorem [Schur]

Let  $A \in \mathbb{C}^{n \times n}$ . There is a unitary  $Q$  such that  $\hat{A} = Q^* A Q$  is upper (lower) triangular.

$$\begin{aligned} AX - XD = E &\longrightarrow Q_A^* A X Q_D - Q_A^* D Q_D = Q_A^* E Q_D \\ &\longrightarrow \hat{A} \hat{X} - \hat{X} \hat{D} = \hat{E}, \quad \hat{X} = Q_A^* X Q_D, \quad \hat{E} = Q_A^* E Q_D. \end{aligned}$$

Step 2: back-substitution

$$\hat{A}_{44} \hat{X}_{44} - \hat{X}_{44} \hat{D}_{44} = \hat{E}_{44}$$

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$$\hat{A}_{33} \hat{X}_{34} - \hat{X}_{34} \hat{D}_{44} = \hat{E}_{34} - \hat{A}_{34} \hat{X}_{44}$$

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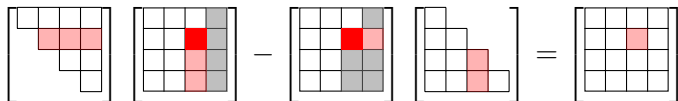
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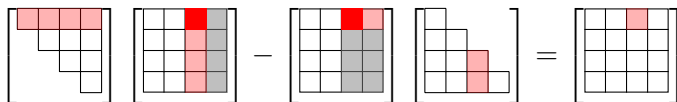
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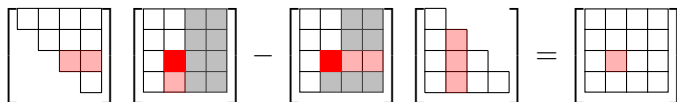
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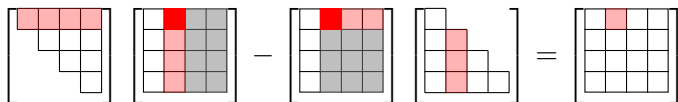
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## Solvability conditions

Equation for  $X_{ij}$  uniquely solvable iff  $\hat{A}_{ij} - \hat{D}_{jj} \neq 0$

**Theorem** [Bartels–Stewart '72]

$AX - XD = C$  well-posed iff  $A$  and  $D$  have no common eigenvalues.

... and a  $\mathcal{O}(n^3)$  algorithm to solve it.

Now, back to systems:

**Example**

$$\begin{cases} AX_1B + CX_2^*D = E \\ FX_1^*G + HX_4I = J \\ KX_3^*L + MX_2^*N = O \end{cases}$$

## Eliminate single-use variables

Suppose  $X_1$  appears only in one equation

$$AX_1^\sigma B + CX_j^\tau D = E.$$

$A$  singular: **Not well posed**: if  $Au = 0$ , we can add multiples of  $uu^*$  to  $X_1$ .

$B$  singular: **Not well posed**, similarly.

$A, B$  invertible: determine uniqueness for the remaining equations, then solve for  $X_1$ .

So we can assume each variable appears **at least exactly** two times.

# Reduce to cycles

## Example

$$\left\{ \begin{array}{l} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{array} \right. \quad \left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right.$$

$Y_1 = X_1,$

## Reduce to cycles

### Example

$$\left\{ \begin{array}{l} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{array} \right. \quad \left\{ \begin{array}{l} B^*Y_1A^* + D^*X_4^*C^* = 0 \\ \\ \\ \\ Y_1 = X_1, \end{array} \right.$$

## Reduce to cycles

### Example

$$\left\{ \begin{array}{l} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{array} \right. \quad \left\{ \begin{array}{l} B^*Y_1A^* + D^*Y_2C^* = 0 \\ \\ \\ \\ \\ \end{array} \right.$$
$$Y_1 = X_1, Y_2 = X_4^*,$$

## Reduce to cycles

### Example

$$\left\{ \begin{array}{l} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{array} \right. \quad \left\{ \begin{array}{l} B^*Y_1A^* + D^*Y_2C^* = 0 \\ \\ \\ \\ \\ \end{array} \right.$$
$$Y_1 = X_1, Y_2 = X_4^*,$$

# Reduce to cycles

## Example

$$\left\{ \begin{array}{l} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{array} \right. \quad \left\{ \begin{array}{l} B^*Y_1A^* + D^*Y_2C^* = 0 \\ KY_2L + IX_5J = 0 \end{array} \right.$$
$$Y_1 = X_1, Y_2 = X_4^*,$$



# Reduce to cycles

## Example

$$\begin{cases} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{cases}$$

$$\begin{cases} B^*Y_1A^* + D^*Y_2C^* = 0 \\ KY_2L + IY_3J = 0 \end{cases}$$

$Y_1 = X_1, Y_2 = X_4^*, Y_3 = X_5$

# Reduce to cycles

## Example

$$\begin{cases} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{cases}$$

$$\begin{cases} B^*Y_1A^* + D^*Y_2C^* = 0 \\ KY_2L + IY_3J = 0 \\ Y_1 = X_1, Y_2 = X_4^*, Y_3 = X_5 \end{cases}$$

# Reduce to cycles

## Example

$$\begin{cases} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{cases}$$

$$\begin{cases} B^*Y_1A^* + D^*Y_2C^* = 0 \\ KY_2L + IY_3J = 0 \\ P^*Y_3O^* + N^*X_1^*M^* = 0 \end{cases}$$
$$Y_1 = X_1, Y_2 = X_4^*, Y_3 = X_5$$

# Reduce to cycles

## Example

$$\begin{cases} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{cases}$$

$$\begin{cases} B^*Y_1A^* + D^*Y_2C^* = 0 \\ KY_2L + IY_3J = 0 \\ P^*Y_3O^* + N^*Y_1^*M^* = 0 \end{cases}$$
$$Y_1 = X_1, Y_2 = X_4^*, Y_3 = X_5$$

# Reduce to cycles

## Example

$$\left\{ \begin{array}{l} AX_1^*B + CX_4D = 0 \\ EX_2^*F + GX_3H = 0 \\ IX_5J + KX_4^*L = 0 \\ MX_1N + OX_5^*P = 0 \\ QX_3^*R + SX_2T = 0 \end{array} \right. \quad \left\{ \begin{array}{l} B^*Y_1A^* + D^*Y_2C^* = 0 \\ KY_2L + IY_3J = 0 \\ P^*Y_3O^* + N^*Y_1^*M^* = 0 \\ \hline EX_2^*F + GX_3H = 0 \\ QX_3^*R + SX_2T = 0 \\ Y_1 = X_1, Y_2 = X_4^*, Y_3 = X_5 \end{array} \right.$$

# Reduction

## Theorem

A Sylvester-like system can be reduced to several disjoint 'cyclic' systems each with at most one nontrivial  $\sigma \in \{1, \star\}$ .

$$\left\{ \begin{array}{l} A_1 Y_1 B_1 - C_1 Y_2 D_1 = 0, \\ A_2 Y_2 B_2 - C_2 Y_3 D_2 = 0, \\ \quad \vdots \\ A_{r-1} Y_{r-1} B_{r-1} - C_{r-1} Y_r D_{r-1} = 0, \\ A_r Y_r B_r - C_r Y_1^\sigma D_r = 0. \end{array} \right.$$

## $\sigma = 1$ : periodic Sylvester equations

If  $\sigma = 1$ , solved in [Byers-Rhee '95].

Idea: generalized Bartels–Stewart algorithm.

Step 1: triangulation

Make  $A_i, C_i$  upper triangular and  $B_i, D_i$  lower triangular.

Theorem (Periodic Schur decomposition, [Bojanczyk-Golub-Van Dooren '92])

Let  $A_1, A_2, \dots, A_r, C_1, C_2, \dots, C_r \in \mathbb{C}^{n \times n}$ . There exist unitary matrices  $Q_1, Q_2, \dots, Q_r, Z_1, Z_2, \dots, Z_r \in \mathbb{C}^{n \times n}$  such that

$$Q_1 A_1 Z_1, Q_2 A_2 Z_2, \dots, Q_r A_r Z_r,$$

$$Q_1 C_1 Z_2, Q_2 C_2 Z_3, \dots, Q_r C_r Z_1$$

are all upper (or: lower) triangular.

## The change of bases

Or: insert orthogonal transformations in  $P = C_r^{-1}A_r C_{r-1}^{-1}A_{r-1} \cdots C_1^{-1}A_1$  to make each factor upper triangular:

$$\underbrace{Z_1^* C_r^{-1} Q_r^*}_{\hat{C}_r^{-1}} \underbrace{Q_r A_r Z_r}_{\hat{A}_r} \underbrace{Z_r^* C_{r-1}^{-1} Q_{r-1}^*}_{\hat{C}_{r-1}^{-1}} \underbrace{Q_{r-1} A_{r-1} Z_{r-1}}_{\hat{A}_{r-1}} \cdots \underbrace{Z_2^* C_1^{-1} Q_1^*}_{\hat{C}_1^{-1}} \underbrace{Q_1 A_1 Z_1}_{\hat{A}_1}.$$

The eigenvalues of  $P$  are  $\lambda_i = \frac{(\hat{A}_r)_{ii}(\hat{A}_{r-1})_{ii} \cdots (\hat{A}_1)_{ii}}{(\hat{C}_r)_{ii}(\hat{C}_{r-1})_{ii} \cdots (\hat{C}_1)_{ii}}$ ,  $i = 1, 2, \dots, n$ .

We can do this formally even if some  $C_k$  are singular ( $\lambda_i$  may be  $\infty$ ).

Omitted in this talk:  $\lambda_i = \frac{0}{0}$ , singular formal product.  
(Spoiler: it's never well-posed.)



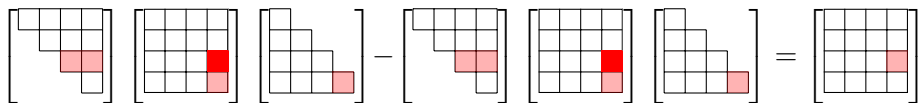
## Step 2: the back-substitution

Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{nn}(B_1)_{nn} & -(C_1)_{nn}(D_1)_{nn} & & & \\
 & (A_2)_{nn}(B_2)_{nn} & -(C_2)_{nn}(D_2)_{nn} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{nn}(B_r)_{nn} \\
 -(C_r)_{nn}(D_r)_{nn} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{nn} \\
 (Y_2)_{nn} \\
 \vdots \\
 (Y_r)_{nn}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{nn}(B_1)_{nn} \cdots (A_r)_{nn}(B_r)_{nn} \neq (C_1)_{nn}(D_1)_{nn} \cdots (C_r)_{nn}(D_r)_{nn}$

## Step 2: the back-substitution

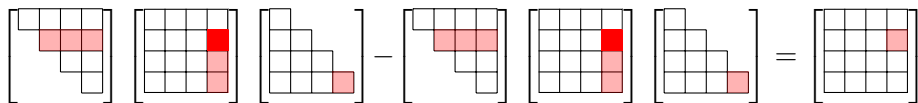


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\ & & \ddots & \ddots & \\ & & & & (A_r)_{ii}(B_r)_{jj} \\ -(C_r)_{jj}(D_r)_{jj} & & & & \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

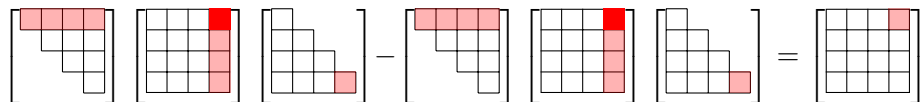


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

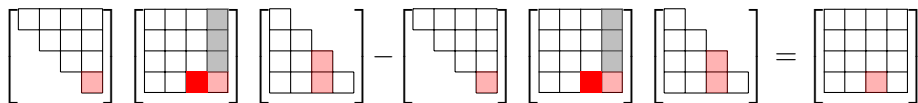


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} & \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

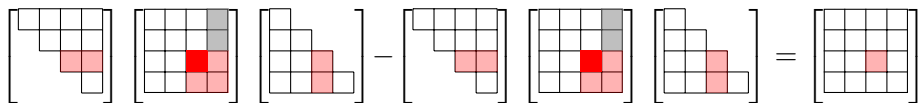


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

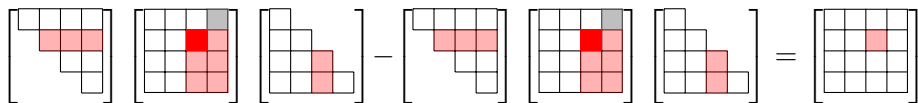


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} & \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

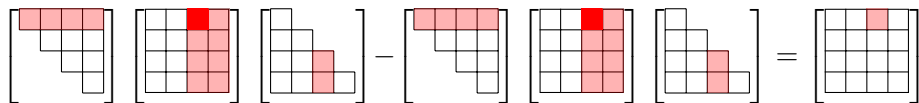


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution



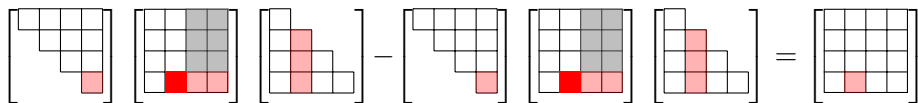
Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$



## Step 2: the back-substitution

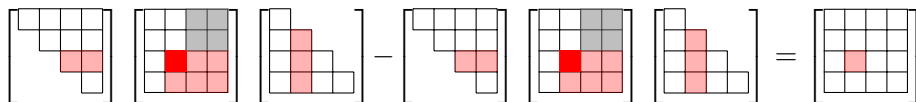


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

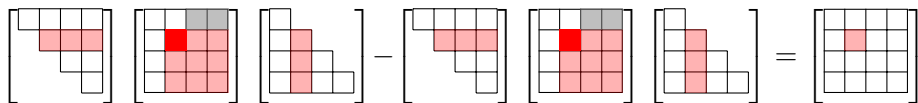


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\ & & \ddots & \ddots & \\ & & & & (A_r)_{ii}(B_r)_{jj} \\ -(C_r)_{jj}(D_r)_{jj} & & & & \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

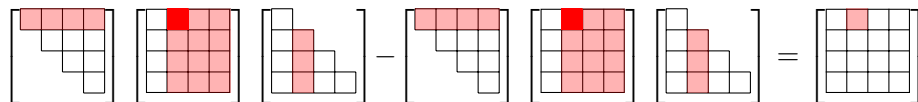


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
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 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

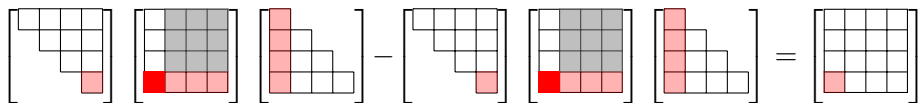


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

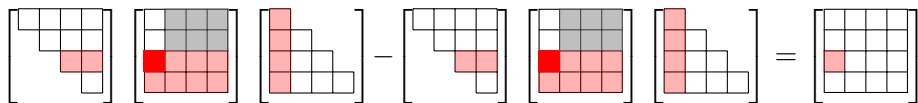


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

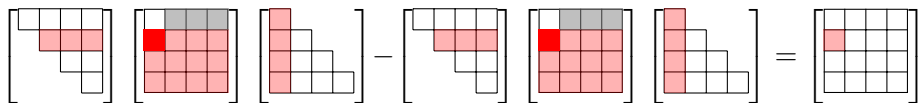


Cyclic bidiagonal system

$$\begin{bmatrix}
 (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\
 & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\
 & & \ddots & \ddots & \\
 & & & & (A_r)_{ii}(B_r)_{jj} \\
 -(C_r)_{jj}(D_r)_{jj} & & & & 
 \end{bmatrix}
 \begin{bmatrix}
 (Y_1)_{ij} \\
 (Y_2)_{ij} \\
 \vdots \\
 (Y_r)_{ij}
 \end{bmatrix}
 = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution

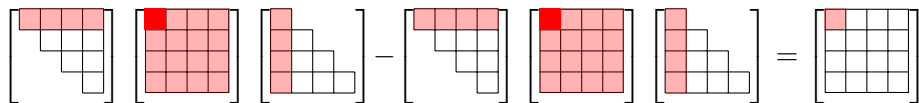


Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ -(C_r)_{jj}(D_r)_{jj} & & & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$

## Step 2: the back-substitution



Cyclic bidiagonal system

$$\begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & & \\ & (A_2)_{ii}(B_2)_{jj} & -(C_2)_{ii}(D_2)_{jj} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ -(C_r)_{jj}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} & \end{bmatrix} \begin{bmatrix} (Y_1)_{ij} \\ (Y_2)_{ij} \\ \vdots \\ (Y_r)_{ij} \end{bmatrix} = R.H.S.$$

Invertible if  $(A_1)_{ii}(B_1)_{jj} \cdots (A_r)_{ii}(B_r)_{jj} \neq (C_1)_{ii}(D_1)_{jj} \cdots (C_r)_{ii}(D_r)_{jj}$



# Well-posedness of systems with $\sigma = 1$

**Theorem** [Byers-Rhee '95]

The cyclic system

$$\begin{cases} A_1 Y_1 B_1 - C_1 Y_2 D_1 = 0, \\ A_2 Y_2 B_2 - C_2 Y_3 D_2 = 0, \\ \vdots \\ A_{r-1} Y_{r-1} B_{r-1} - C_{r-1} Y_r D_{r-1} = 0, \\ A_r Y_r B_r - C_r Y_1 D_r = 0. \end{cases}$$

is well posed **iff** the formal products

$$\begin{aligned} R &= C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1 \\ S &= D_r B_r^{-1} D_{r-1} B_{r-1}^{-1} \cdots D_1 B_1^{-1} \end{aligned}$$

have no common eigenvalues.

## $\sigma = \star$ : periodic $\star$ -Sylvester systems

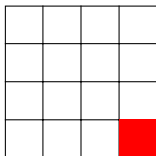
Step 1: compute periodic Schur form of

$$S^{-\star}R = D_r^{-\star}B_r^{\star}D_{r-1}^{-\star}B_{r-1}^{\star}\cdots D_1^{-\star}B_1^{\star}C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1.$$

Step 2: block back-substitution: solve a  $2r \times 2r$  system for

$$(Y_1)_{ij}, (Y_2)_{ij}, \dots, (Y_r)_{ij}, (Y_1)_{ji}, (Y_2)_{ji}, \dots, (Y_r)_{ji}$$

simultaneously.



## $\sigma = \star$ : periodic $\star$ -Sylvester systems

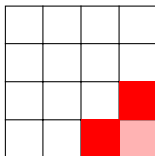
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$$S^{-\star}R = D_r^{-\star}B_r^{\star}D_{r-1}^{-\star}B_{r-1}^{\star}\cdots D_1^{-\star}B_1^{\star}C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1.$$

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simultaneously.



## $\sigma = \star$ : periodic $\star$ -Sylvester systems

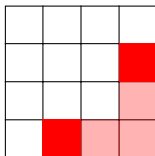
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$$S^{-\star}R = D_r^{-\star}B_r^{\star}D_{r-1}^{-\star}B_{r-1}^{\star}\cdots D_1^{-\star}B_1^{\star}C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1.$$

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simultaneously.



## $\sigma = \star$ : periodic $\star$ -Sylvester systems

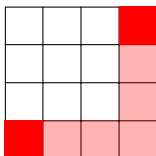
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## $\sigma = \star$ : periodic $\star$ -Sylvester systems

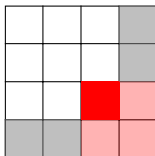
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simultaneously.



## $\sigma = \star$ : periodic $\star$ -Sylvester systems

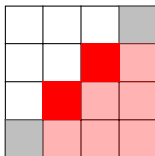
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simultaneously.



## $\sigma = \star$ : periodic $\star$ -Sylvester systems

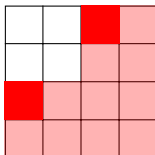
Step 1: compute periodic Schur form of

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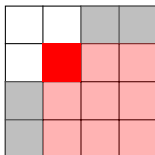
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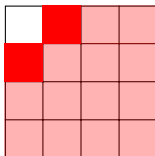
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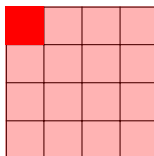
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# The well-posedness conditions

Reciprocal-free conditions:

## Theorem

Let  $\lambda_i$ ,  $i = 1, 2, \dots, n$  be the eigenvalues of the formal matrix product  $S^{-\star}R$ . A cyclic  $\star$ -Sylvester system is well-posed iff

$$\boxed{\star = \star} : \lambda_i \bar{\lambda}_j \neq 1 \quad \forall i, j:$$

$$\lambda_i \bar{\lambda}_i \neq 1 \quad \forall i \quad (\text{from the diagonal blocks})$$

$$\lambda_i \bar{\lambda}_j \neq 1 \quad \forall i \neq j \quad (\text{from the off-diagonal blocks})$$

---

$$\boxed{\star = \top} : \lambda_i \lambda_j \neq 1 \quad \forall i, j, \text{ but } \lambda = -1 \text{ may appear once:}$$

$$\lambda_i \neq 1 \quad \forall i \quad (\text{from the diagonal blocks})$$

$$\lambda_i \lambda_j \neq 1 \quad \forall i \neq j \quad (\text{from the off-diagonal blocks})$$

## Another approach: duplicate the system

### Example

$$A_1 Y_1 B_1 - C_1 Y_2 D_1 = E_1$$

$$A_2 Y_2 B_2 - C_2 Y_3 D_2 = E_2$$

$$A_3 Y_3 B_3 - C_3 Y_1^* D_3 = E_3$$

## Another approach: duplicate the system

### Example

$$A_1 Y_1 B_1 - C_1 Y_2 D_1 = E_1$$

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$$A_3 Y_3 B_3 - C_3 Y_1^* D_3 = E_3$$

$$B_1^* Y_1^* A_1^* - D_1^* Y_2^* C_1^* = E_1^*$$

$$B_2^* Y_2^* A_2^* - D_2^* Y_3^* C_2^* = E_2^*$$

$$B_3^* Y_3^* A_3^* - D_3^* Y_1 C_3^* = E_3^*$$

## Another approach: duplicate the system

### Example

$$A_1 Y_1 B_1 - C_1 Y_2 D_1 = E_1$$

$$A_2 Y_2 B_2 - C_2 Y_3 D_2 = E_2$$

$$A_3 Y_3 B_3 - C_3 Z_1 D_3 = E_3$$

$$B_1^* Z_1 A_1^* - D_1^* Z_2 C_1^* = E_1^*$$

$$B_2^* Z_2 A_2^* - D_2^* Z_3 C_2^* = E_2^*$$

$$B_3^* Z_3 A_3^* - D_3^* Y_1 C_3^* = E_3^*$$

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$$B_3^* Z_3 A_3^* - D_3^* Y_1 C_3^* = E_3^*$$

### Lemma (only for $\star = \ast$ )

*The 'duplicated' Sylvester system has a unique solution iff the original  $\star$ -Sylvester system has unique solution.*

Solvability condition for the duplicated system:  $S^{-\star}R$  and  $R^{-\star}S$  have no common eigenvalues.

Gives immediately the well-posedness condition  $\lambda_i \bar{\lambda}_j \neq 1 \forall i, j$ .



# The duplication lemma

Lemma (only for  $\star = *$ )

The 'duplicated' Sylvester system has a unique solution iff the original  $\star$ -Sylvester system has unique solution.

Nontrivial part: we need to prove that if

$$\begin{array}{ll} A_1 Y_1 B_1 - C_1 Y_2 D_1 = 0 & B_1^* Z_1 A_1^* - D_1^* Z_2 C_1^* = 0 \\ A_2 Y_2 B_2 - C_2 Y_3 D_2 = 0 & B_2^* Z_2 A_2^* - D_2^* Z_3 C_2^* = 0 \\ A_3 Y_3 B_3 - C_3 Z_1 D_3 = 0 & B_3^* Z_3 A_3^* - D_3^* Y_1 C_3^* = 0 \end{array}$$

has a nonzero solution, then it has one with **property (\*)**:  $Z_i = Y_i^* \forall i$ .

$(Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*)$  is another solution.

By linearity,  $(Y_1 + Z_1^*, Y_2 + Z_2^*, Y_3 + Z_3^*, Z_1 + Y_1^*, Z_2 + Y_2^*, Z_3 + Y_3^*)$  is a solution, and has property (\*).

# The duplication lemma

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has a nonzero solution, then it has one with **property (\*)**:  $Z_i = Y_i^* \forall i$ .

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By linearity,  $(Y_1 + Z_1^*, Y_2 + Z_2^*, Y_3 + Z_3^*, Z_1 + Y_1^*, Z_2 + Y_2^*, Z_3 + Y_3^*)$  is a solution, and has property (\*). **But it might be zero!**

Let's try again: By linearity,

$(i(Y_1 - Z_1^*), i(Y_2 - Z_2^*), i(Y_3 - Z_3^*), i(Z_1 - Y_1^*), i(Z_2 - Y_2^*), i(Z_3 - Y_3^*))$  is a solution, and has property (\*).

## Duplication lemma - counterexample

We needed the fact that  $(aX)^* = \bar{a}X^*$ . For  $\top$ , it doesn't work.

### Counterexample

$$2y + 2y = 0$$

has unique solution,

$$\begin{cases} 2y + 2z = 0 \\ 2z + 2y = 0 \end{cases}$$

does not.

# Recap

- Well-posedness of  $\star$ -Sylvester systems reduced to cyclic case.
- Depends only on eigenvalues of certain formal products.
- Bartels–Stewart-like algorithm (non-structurally backward stable).
- Duplication algorithm ( $\star = *$ ) (non-structurally backward stable).
- Same ideas work for systems with mixed  $\top$ ,  $*$  and  $-$  symbols.
- More than two terms per equation: **more involved** already for  $r = 1$ .
- Non-square case: **more involved** already for  $r = 1$ , see [De Terán, Iannazzo, P, Robol, Arxiv '16].

# The non-square case, $r = 1$

**Theorem** ([De Terán, Iannazzo, Poloni, Robol '16 Arxiv])

$AXB + CX^*D = E$  well-posed iff the following conditions hold:

- Either  $(A$  and  $B)$  or  $(C$  and  $D)$  are square
- The **smaller** of these two square coefficients is invertible
- $Q(\lambda) = \begin{bmatrix} \lambda D^* & B^* \\ A & \lambda C \end{bmatrix}$  is regular with 'reciprocal-free' eigenvalues.

$Q(\lambda)$  'replaces'  $D^{-*}B^*C^{-1}A$  (tricky to define: [Granat Kågström Kressner '07]).

Eigenvalues alone **not sufficient** to determine well-posedness:

## Counterexample

$$Q_1(\lambda) = \left[ \begin{array}{cc|c} \lambda & 0 & 0 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{array} \right], \quad Q_2(\lambda) = \left[ \begin{array}{cc|c} \lambda & 0 & 1 \\ 0 & 0 & \lambda \\ 0 & 1 & 0 \end{array} \right].$$

## Why only two terms?

If we can solve systems with 3 terms, we can solve equations with arbitrarily many, by adding variables.

### Example

$$AXB + CXD + EXF + GXH + IXJ + KXL = M$$



$$AXB + CXD + Y = 0,$$

$$EXF + GXH + Z = 0,$$

$$IXJ + KXL + W = 0,$$

$$Y + Z + W = M.$$

Related to a big open problem (what is the complexity  $\mathcal{O}(n^\tau)$  of matrix multiplication?)

### When do they exist? — I

When is a system of Sylvester-like matrix equations solvable?

**F. De Terán**, R. Izquierdo, L. Rodas

Mathematical Collection, 2019, 16 March 2020

Each equation is in the form:

$$AX + XA = C, \quad X \in \mathbb{R}^{n \times n}$$

where  $A, C \in \mathbb{R}^{n \times n}$  (with  $A$  constant along all diagonals).

• Solvability:  $\text{rank}(A) = \text{rank}(C)$ . Test of them in each equation.

• Rank:  $\text{rank}(A) = \text{rank}(C)$  is a necessary condition for solvability.

### When do they exist? — II

Decoupling system of equations:

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

Is there a change of basis that does this?

$$P^{-1} A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad P^{-1} C P = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

Then the system decouples into  $n$  independent equations.

### One equation: the Bartlett-Stewart algorithm

Given  $A, C \in \mathbb{R}^{n \times n}$ .

Step 1: reduce to triangular form.

$$AX + XA = C \implies (A + \lambda I)X + X(A - \lambda I) = C - \lambda X$$

Step 2: back substitution.

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

### The change of basis

Do linear orthogonal transformations  $P = [v_1 \dots v_n]$  exist such that  $P^{-1}AP$  is upper triangular?

The eigenvalues of  $P^{-1}AP$  are  $\lambda_1, \dots, \lambda_n$ .

We use this family since  $\lambda_i \neq \lambda_j$  for  $i \neq j$ .

Observe in this table:  $\lambda_i \neq \lambda_j$  implies  $\text{rank}(A) = \text{rank}(C)$ .

### Another approach: duplicate the system

Consider:

$$\begin{cases} AX + XA = C \\ AX + XA = C \\ \vdots \\ AX + XA = C \end{cases}$$

The "duplicate" system has a unique solution if the original Sylvester system has unique solution.

Normal form:  $\text{rank}(A) = \text{rank}(C)$ .

Has a unique solution:  $\text{rank}(A) = \text{rank}(C)$ .

By theory:  $\text{rank}(A) = \text{rank}(C) \implies \text{rank}(A) = \text{rank}(C)$ .

### When do they exist? — III

Example:

$$\begin{cases} AX + XA = C \\ AX + XA = C \\ \vdots \\ AX + XA = C \end{cases}$$

Each equation is in the form:

$$AX + XA = C, \quad X \in \mathbb{R}^{n \times n}$$

where  $A, C \in \mathbb{R}^{n \times n}$  (with  $A$  constant along all diagonals).

• Solvability:  $\text{rank}(A) = \text{rank}(C)$ . Test of them in each equation.

• Rank:  $\text{rank}(A) = \text{rank}(C)$  is a necessary condition for solvability.

### When do they exist? — IV

Decoupling system of equations of the form  $AX + XA = C$ :

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

Is there a change of basis that does this?

$$P^{-1} A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad P^{-1} C P = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

Then the system decouples into  $n$  independent equations.

### Solvability condition

Equation for  $X_n$  uniquely solvable if  $\lambda_n \neq \lambda_n$ .

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

Step 2: the back-substitution

Cyclic algorithm:

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

Invertible:  $(A_1 + \lambda_1 I)X_1 = C_1 - \lambda_1 X_1 \implies (A_1 + \lambda_1 I)X_1 = C_1 - \lambda_1 X_1$ .

### Step 2: the back-substitution

Cyclic algorithm:

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

Invertible:  $(A_1 + \lambda_1 I)X_1 = C_1 - \lambda_1 X_1 \implies (A_1 + \lambda_1 I)X_1 = C_1 - \lambda_1 X_1$ .

### The duplication lemma

Lemma (Sylvester):

The "duplicate" system has a unique solution if the original Sylvester system has unique solution.

Normal form:  $\text{rank}(A) = \text{rank}(C)$ .

Has a unique solution:  $\text{rank}(A) = \text{rank}(C)$ .

By theory:  $\text{rank}(A) = \text{rank}(C) \implies \text{rank}(A) = \text{rank}(C)$ .

### When do they exist? — V

Stability of dynamical systems:

System (Sylvester stability):

$$AX + XA = C, \quad X \in \mathbb{R}^{n \times n}$$

where  $A, C \in \mathbb{R}^{n \times n}$  (with  $A$  constant along all diagonals).

• Solvability:  $\text{rank}(A) = \text{rank}(C)$ . Test of them in each equation.

• Rank:  $\text{rank}(A) = \text{rank}(C)$  is a necessary condition for solvability.

### Vectorization

$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$  is a linear system:  $AX + XA = C$ .

Example:

$$AX + XA = C \implies \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \text{vec}(X) \\ \text{vec}(X) \end{bmatrix} = \text{vec}(C)$$

where  $\text{vec}(X) = [x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{n1}, \dots, x_{nn}]^T$ .

### Eliminate single-use variables

Suppose  $X_n$  appears only in one equation:

$$AX_n + X_n A = C_n$$

It appears **not** in other equations. If  $\lambda_n \neq \lambda_n$ , we can solve for  $X_n$  and substitute it into the remaining equations.

For us to remove such variables appears **not** useful here.

### Well-posedness of systems with $n = 1$

The cyclic system:

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

is well-posed if  $\lambda_i \neq \lambda_j$  for  $i \neq j$ .

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

### Duplication lemma - counterexample

We consider the case  $\lambda_1 = \lambda_2 = \dots = \lambda_n$  is not a rank.

Counterexample:

$$\begin{cases} AX + XA = C \\ AX + XA = C \\ \vdots \\ AX + XA = C \end{cases}$$

where  $A, C \in \mathbb{R}^{n \times n}$  (with  $A$  constant along all diagonals).

### When do they exist? — VI

Decoupling system of equations:

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

Is there a change of basis that does this?

$$P^{-1} A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad P^{-1} C P = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

Then the system decouples into  $n$  independent equations.

### Well-posedness

System of generalized Sylvester equations:  $AX + XA = C$ .

Rank condition:

$$\text{rank}(A) = \text{rank}(C)$$

Is there a system uniquely solvable for  $X$ ?

How can we use the solution?

Algorithm used for numerical problems: because it means the answer is stable under small perturbations.

Four conditions: number of equations = number of unknowns.

### Reduce to cyclic

Example:

$$\begin{cases} AX + XA = C \\ AX + XA = C \\ \vdots \\ AX + XA = C \end{cases}$$

where  $A, C \in \mathbb{R}^{n \times n}$  (with  $A$  constant along all diagonals).

### $n = n$ : periodic Sylvester systems

Step 1: compute periodic factor form:

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_n \end{bmatrix}$$

Step 2: block back-substitution using a  $2n \times 2n$  system for:

$$\begin{cases} A_1 X_1 + X_1 A_1 = C_1 \\ A_2 X_2 + X_2 A_2 = C_2 \\ \vdots \\ A_n X_n + X_n A_n = C_n \end{cases}$$

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

### Rank

Well-posedness of a Sylvester system reduced to cyclic case:

- Depends only on eigenvalues of certain formal products.
- Rank:  $\text{rank}(A) = \text{rank}(C)$ .
- Multiplied algorithm:  $\text{rank}(A) = \text{rank}(C)$ .
- Some cases need the system with  $n = 1$  well-posed.
- More than two linear equations: more involved already for  $n = 1$ .
- Non-square case: **not** solvable already for  $n = 1$ .

### Rank-epi condition

Rank condition:  $\text{rank}(A) = \text{rank}(C)$ .

How can we use the solution?

Algorithm used for numerical problems: because it means the answer is stable under small perturbations.

Four conditions: number of equations = number of unknowns.

### $n = 1$ : periodic Sylvester equations

Step 1: integrate:

$$\begin{cases} AX + XA = C \\ AX + XA = C \\ \vdots \\ AX + XA = C \end{cases}$$

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

### The well-posedness condition

Rank condition:  $\text{rank}(A) = \text{rank}(C)$ .

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

### The non-square case: $n = 1$

Rank condition:  $\text{rank}(A) = \text{rank}(C)$ .

Rank:  $\text{rank}(A) = \text{rank}(C)$ .

Thanks for your attention!  
Questions?