Estimating Econometric Models through Matrix Equations

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VARMA(1,1) models

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$$x_t - \Phi x_{t-1} = u_t - \Theta u_{t-1}$$

x_t = observed variable $\in \mathbb{R}^d$
u_t = white noise (enough to assume uncorrelated) $\in \mathbb{R}^d$
$\Phi, \Theta \in \mathbb{R}^{d\times d}$, $\rho(\Phi) < 1$, $\rho(\Theta) < 1$

Many known models to simulate volatility reduce to VARMA(1,1):

- GARCH(1,1)
- Multivariate stochastic volatility models

Estimating VARMAs

Problem

Given enough observations (x_t) generated by a VARMA, determine parameters Φ, Θ

A common choice is QML (quasi-maximum-likelihood):

1. Assume u_t Gaussian independent
2. Given guesses $\hat{\Phi}, \hat{\Theta}$, compute likelihood $\ell(\hat{\Phi}, \hat{\Theta})$ of generating the given time series
3. Feed $\ell(\cdot, \cdot)$ into a black-box minimization procedure (e.g., Matlab's `fminunc`)

Problems with QML

1. Costly: each function evaluation costs $O(nd^3)$, with $n =$ length of time series. Hundreds or thousands required
2. Black-box: difficult to implement and tweak, and understand what's going on.
3. No convergence guarantees, non-convex optimization problem in many variables
4. Hey doc, what if our u_t isn't Gaussian independent?

Our attempt

Moments estimator: determine Φ, Θ as a function of the autocovariances

$$M_k = \mathbb{E}_t \left[x_t x_{t+k}^T \right]$$

($\mathbb{E}_t [\cdot] :=$ stationary limit mean)

The M_k contain cov's among all variables of the time series, e.g., ($d = 3$)

$$x_t = \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix} \Rightarrow M_k = \begin{bmatrix} \mathbb{E}_t [a_t a_{t+k}] & \mathbb{E}_t [a_t b_{t+k}] & \mathbb{E}_t [a_t c_{t+k}] \\ \mathbb{E}_t [b_t a_{t+k}] & \mathbb{E}_t [b_t b_{t+k}] & \mathbb{E}_t [b_t c_{t+k}] \\ \mathbb{E}_t [c_t a_{t+k}] & \mathbb{E}_t [c_t b_{t+k}] & \mathbb{E}_t [c_t c_{t+k}] \end{bmatrix}$$

Moment estimator

We will show $(\Phi, \Theta) = f(M_0, M_1, M_2)$

GMM estimator

1. Compute sample $\hat{M}_k = \frac{1}{n} \sum x_t x_{t+k}^T$
2. Get $(\hat{\Phi}, \hat{\Theta}) = f(\hat{M}_0, \hat{M}_1, \hat{M}_2)$

- Very fast: working only with $d \times d$ matrices, no dependence on n (after computing moments)
- Asymptotically consistent and normal, under suitable conditions
- In simulated experiments, not as accurate as QML, but good as initial value

Already known for univariate GARCH; generalization requires some linear algebra machinery

Yule-Walker results

The parameter Φ is easy to obtain:

Theorem

$\Phi = M_{k+1} M_k^{-1}$ for each $k \geq 1$

In particular $\Phi = M_2 M_1^{-1}$

Proof:

$$\underbrace{x_{t-k-1}(x_t - \Phi x_{t-1})^T}_{= M_{k+1} - \Phi M_k} = \underbrace{x_{t-k-1}(u_t - \Theta u_{t-1})^T}_{=0 \text{ if } k \geq 1}$$

Estimating Θ

Let $r_t = x_t - \Phi x_{t-1} = u_t - \Theta u_{t-1}$, $Y := \mathbb{E}\left[u_t u_t^T\right]$

$$\Gamma_0 := \mathbb{E}_t\left[r_t r_t^T\right] = M_0 - \Phi M_1^T - M_1 \Phi^T + \Phi M_0 \Phi^T = Y + \Theta Y \Theta^T$$
$$\Gamma_1 := \mathbb{E}_t\left[r_t r_{t+1}^T\right] = M_1 - \Phi M_0 = -\Theta Y$$

Blue expressions allow us to compute Γ_0, Γ_1.
Use them + red expressions to decouple equations for Y, $X = \Theta^T$

$$\Gamma_0 = Y + \Gamma_1 Y^{-1} \Gamma_1^T$$
$$\Gamma_1^T + \Gamma_0 X + \Gamma_1 X^2 = 0$$

Existence and unicity

$$\Gamma_0 = Y + \Gamma_1 Y^{-1} \Gamma_1^T$$
$$\Gamma_1^T + \Gamma_0 X + \Gamma_1 X^2 = 0$$

Studied by matrix equation people (like me)

Existence and unicity

- Solution exists if $Q(\lambda) := \Gamma_1^T \lambda^{-1} + \Gamma_0 + \Gamma_1 \lambda$ is such that $Q(\lambda) > 0$ for each λ on unit circle
- Solution unique if we ask $Y > 0$, $\rho(X) < 1$ (as was assumed)

Of course, if the model is well-posed, there must be a solution...

How to solve matrix equations

Let us focus on $\Gamma_1^T + \Gamma_0 X + \Gamma_1 X^2 = 0$
Solution resembles a lot linear recurrence theory

Generalized eigenvalues/Vectors of the problem:

$$(\lambda, v) \quad \text{s.t.} \quad (\Gamma_1^T + \Gamma_0 \lambda + \Gamma_1 \lambda^2) v = 0$$

(there are $2d$ of them!)

Theorem

Solutions can be eigendecomposed as $X = VDV^{-1}$
V contains d of the $2d$ eigenvectors of the problem, D diagonal with eigenvalues

Generalized eigenvalues

Companion matrix

Eigenvalues/eigenvectors are in 1:1 correspondence with those of the linearization matrix

$$\begin{bmatrix} 0 & I_d \\ -\Gamma_1^{-1}\Gamma_1^T & -\Gamma_1^{-1}\Gamma_0 \end{bmatrix}$$

Matrix version of the "companion matrix" for polynomials

Palindromic matrix polynomials

Due to the structure in $\Gamma_1^T + \Gamma_0\lambda + \Gamma_1\lambda^2$, $\Gamma_0 = \Gamma_0^T$, eigenvalues come in pairs (λ, λ^{-1})

Good for us — we needed d of them with $|d| < 1$!

To sum up

1. Get sample moments M_0, M_1, M_2
2. Get $\Phi = M_2 M_1^{-1}$
3. Compute Γ_0, Γ_1
4. Get eigenvalues/vectors of $\begin{bmatrix} 0 & I_d \\ -\Gamma_1^{-1}\Gamma_1^T & -\Gamma_1^{-1}\Gamma_0 \end{bmatrix}$
5. Take those with λ inside the unit circle
6. Assemble $\Theta^T = X = VDV^{-1}$

What can go wrong

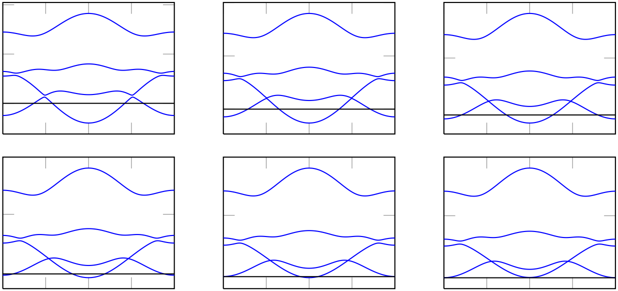
Sometimes, no stationary/invertible model with autocovariances $\hat{M}_i$

Existence and unicity

- Solution X, Y exists if $Q(\lambda) := \Gamma_1^T \lambda^{-1} + \Gamma_0 + \Gamma_1 \lambda$ is such that $Q(\lambda) > 0$ for each λ on unit circle

Must hold with exact Γ_i, but sample $\hat{\Gamma}_i$ might give inconsistent data

$\Rightarrow$ Perturb M_0, M_1, M_2 to ensure solvability!



Some results

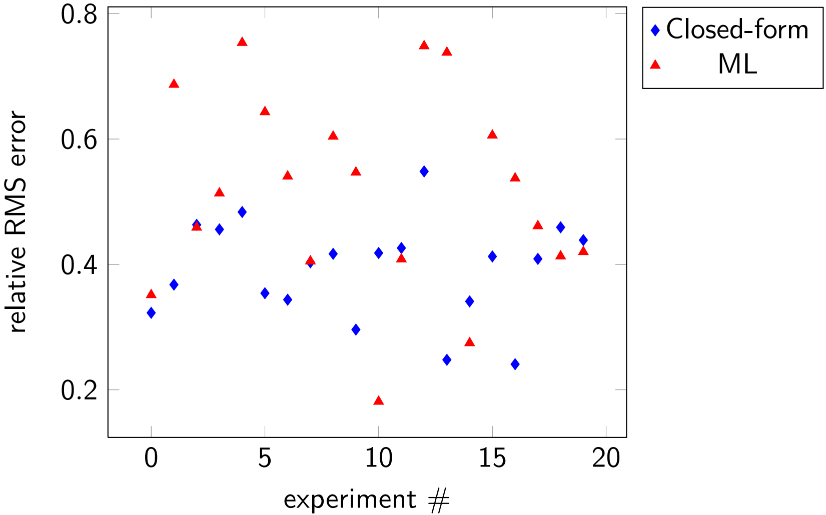
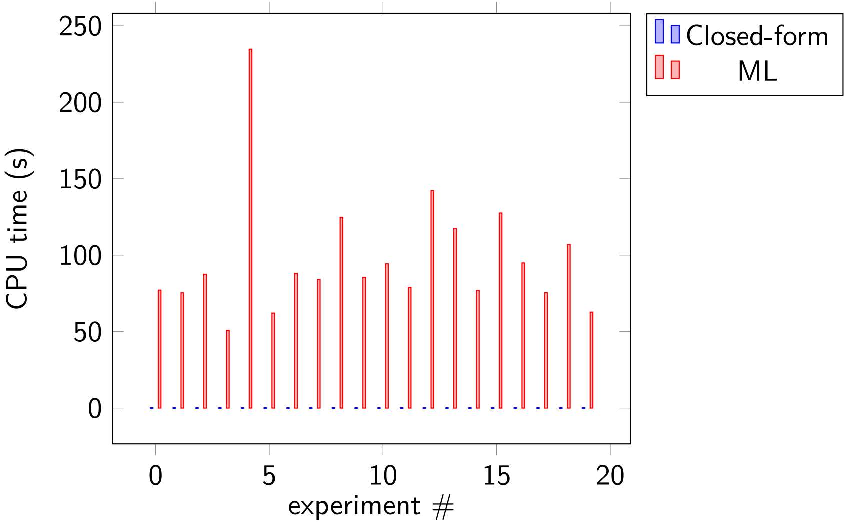


Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.6$, $n = 1000$

Some results



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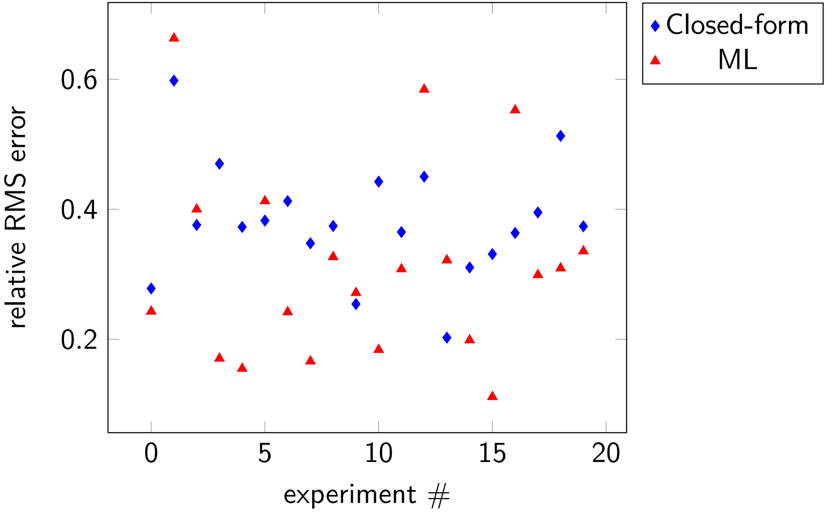

Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.7$, $n = 1000$

Some results

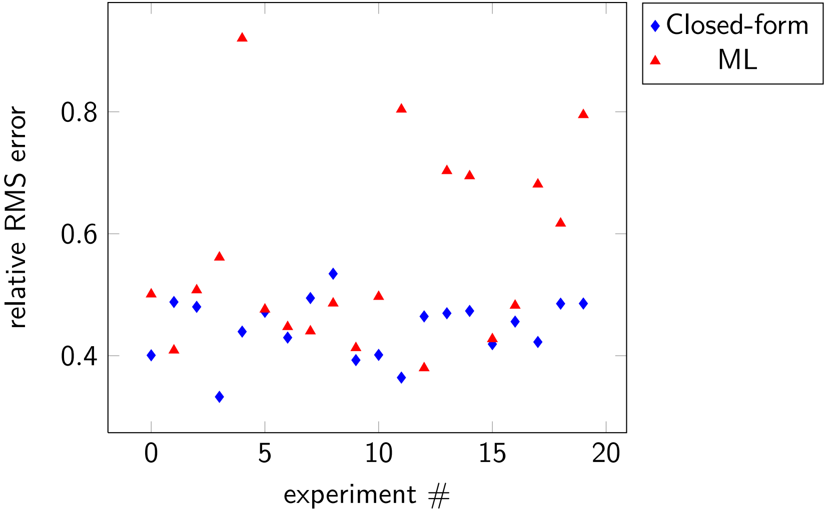

Figure: Diagonal GARCH, $d = 3$, $\rho(\Theta) = 0.6$, $n = 1000$

Some results

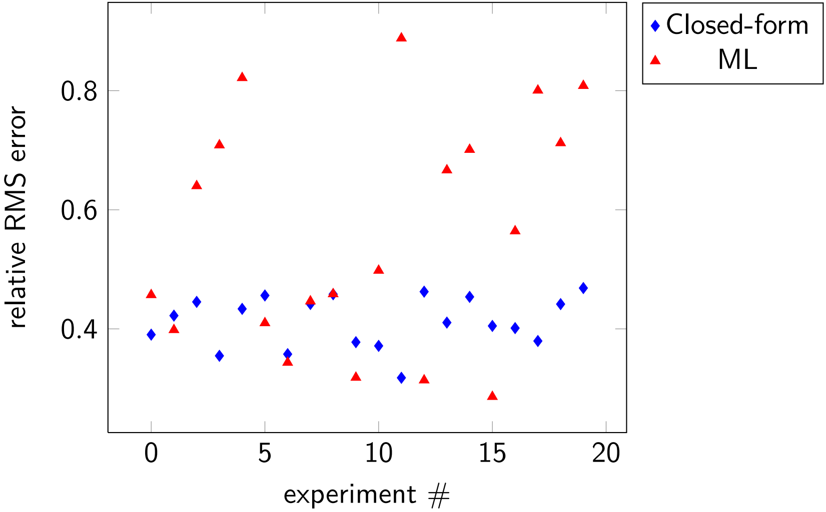

Figure: Diagonal GARCH, $d = 3$, $\rho(\Theta) = 0.6$, $n = 1000$, larger off-diagonal elements

Some results

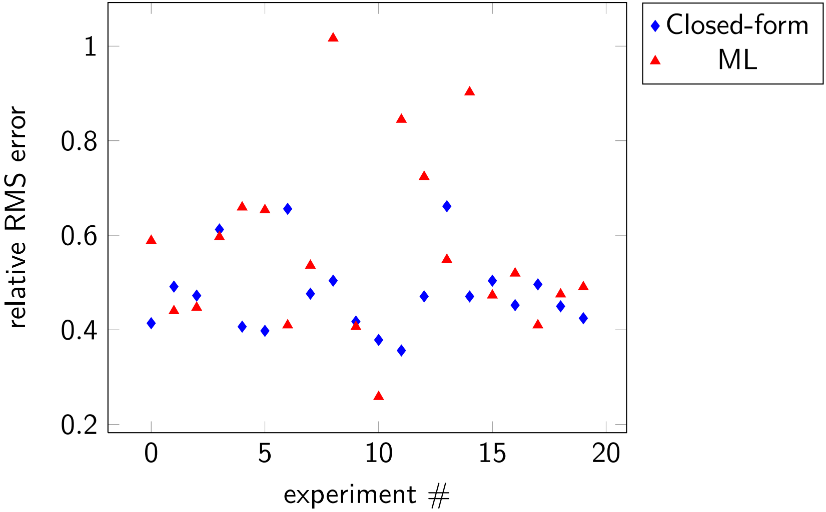

Figure: Diagonal GARCH, $d = 3$, $\rho(\Theta) = 0.6$, $n = 500$

Some results

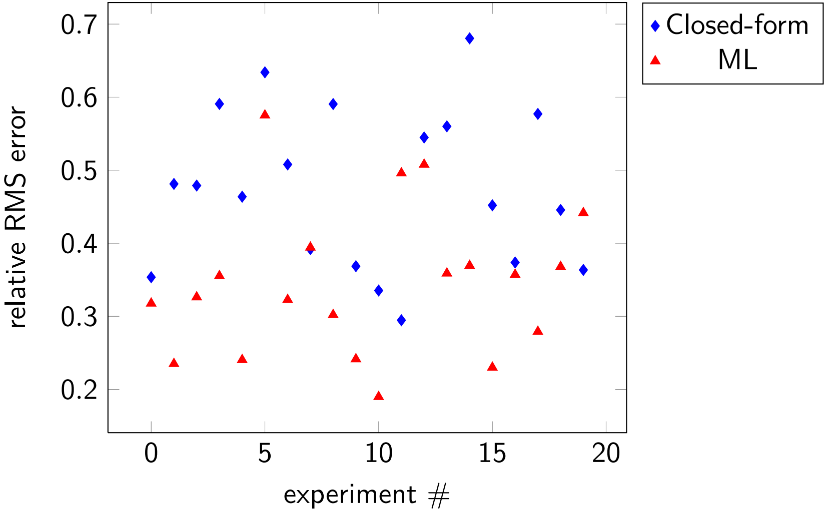

Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.7$, $n = 500$

Some results

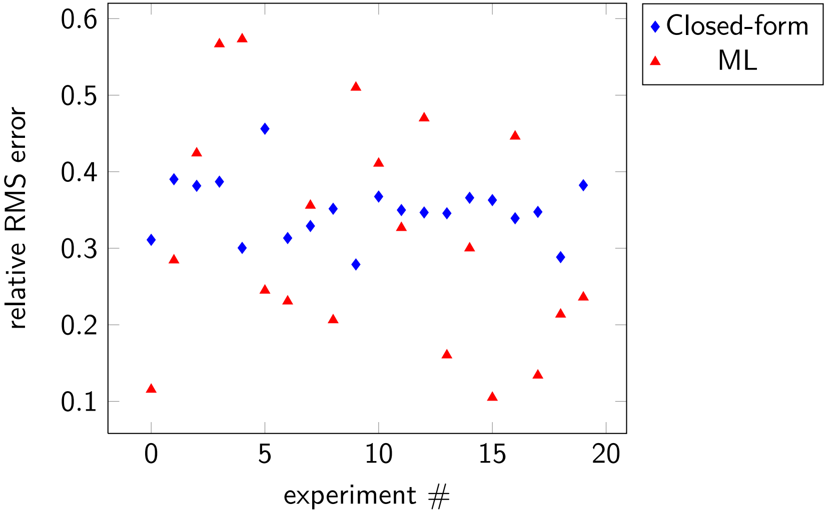

Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.6$, $n = 5000$

Some results

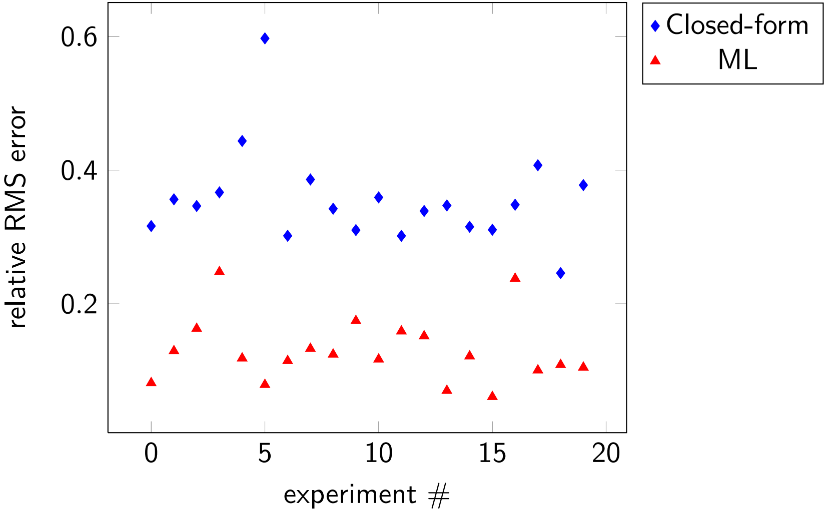

Figure: Diagonal GARCH, $d = 2$, $\rho(\Theta) = 0.7$, $n = 5000$

Possible improvements

- Improve solvability enforcement (work on Θ and Φ at the same time)
- Combine with an iterative ML-like optimization
  e.g., GLS (generalized least squares) for GARCH?
- More intensive testing & applications

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Thanks for your attention!