Triplet representations for matrix equations in queuing theory

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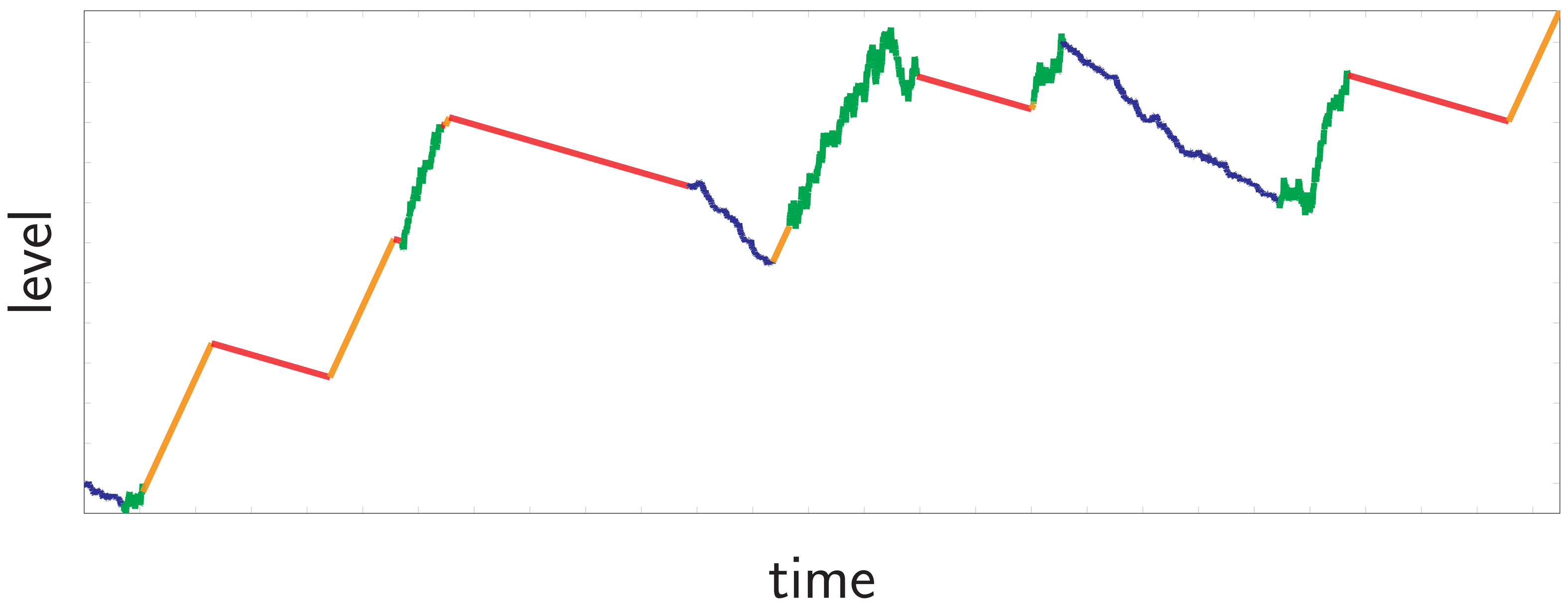
Our problem

Markov-modulated fluid queues and Brownian motion: liquid in an infinite buffer; in- and out-flow (and possibly BM variance) depend on environment state (continuous-time Markov chain)



Stationary measure for level x: $p(x) \in \mathbb{R}^{1\times n}$ satisfying a BVP

$$\ddot{p}V - \dot{p}D + Q = 0 \quad + \text{boundary cond'ns at 0 and } \infty$$

V: diagonal with $v_{ii} \geq 0$
D: diagonal with mixed-sign d_{ii}
Q: continuous-time Markov chain, $Q\underline{\mathbf{1}} = 0$, offdiag$(Q) \geqslant 0$

Key to find it: left stable invariant pair, $X^2UV - XUD + UQ = 0$
U (hopefully $\geqslant 0$) "projection", X square containing eigenvalues of the problem in the (open) left half-plane

Accuracy goal

Our aim solving the problem with componentwise accuracy:

$$|M - \tilde{M}| \leqslant \varepsilon M \quad \text{for computed quantity } M \geqslant 0$$

Inequality and $|\cdot|$ to hold entrywise, even on very small entries
Problem Subtractive cancellation $\to$ loss of significant digits
Solution Avoid all the subtractions!

Triplet representations

An M-matrix A can be recovered from:
- its off-diagonal part, offdiag(A), and
- $v > 0$, $w \geqslant 0$ such that $Av = w$

Example 1

$$\begin{bmatrix} 1 & -1 \\ -1 & 1+\varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$

Values in red (offdiag$(A), v, w$) uniquely determine the matrix

With a triplet representation, one can run subtraction-free Gaussian elimination on A (*GTH-like algorithm*): solves systems with componentwise error $O(n^3\mathbf{u})$. No condition number!

Matrix entries — ill-conditioned $\kappa_{cw}(A)\mathbf{u}$ — Triplet repres — well-conditioned $O(n^3\mathbf{u})$ — A^{-1}

The first-order case

Case $V = 0$ more studied. Formulated equivalently as nonsymmetric algebraic Riccati equation. Best class of algorithms: doubling
[Xue, Xu, Li] gave perturbation bound, highlighted need for triplets

First-order case: doubling algorithm

Idea Compute stable modes from $\lim_{t\to\infty}\exp(At)$, via a matrix-pencil version of scaling and squaring

1. First approximation via Padé ("continuous to discrete") $\exp(tx) \approx \mathcal{C}(x) = (1+\beta x)(1-\alpha x)^{-1}$
2. Squaring, representing intermediate matrices implicitly

What we add to this case:

Triplets

Explicit triplets for the inversions $\implies$ subtraction-free algorithm

$$\text{triplet}(I - G_kH_k) = (\text{offdiag}(G_kH_k), \underline{\mathbf{1}}, E_k\underline{\mathbf{1}} + G_kF_k\underline{\mathbf{1}})$$
$$\text{triplet}(I - H_kG_k) = (\text{offdiag}(H_kG_k), \underline{\mathbf{1}}, F_k\underline{\mathbf{1}} + H_kE_k\underline{\mathbf{1}})$$

No need to get them back from the matrix entries

Error analysis

Componentwise accurate $U \geqslant 0$ and triplet representation for X:

$$|U - \tilde{U}| \leqslant c\mathbf{u}U, \qquad \text{same for } X$$

Coefficient c grows:
- linearly with $1 - \rho(\mathcal{C}(X))$ (distance to instability)
- cubically (upper bound, linear in practice) with dimension n

Second-order case: cyclic reduction

Base algorithm (from [Latouche, Nguyen]):
1. continuous-to-discrete transformation $y = \mathcal{C}(x)$
2. Cyclic reduction (CR) – classical doubling-type algorithm for quadratic problems

Problems
- Not subtraction-free — signs are simply wrong for that
- Infinite eigenvalues from zeros in V complicate convergence

Solution: shift technique/order reduction. With correct choices and parameters, fixes both issues at the same time
Infinite eigenvalues deflated automatically

$$P(x) = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x^2 - \begin{bmatrix} * & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{bmatrix} x + \begin{bmatrix} * & + & + \\ + & * & + \\ + & + & * \end{bmatrix}$$

$$\widehat{P}(x) = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} x^2 - \begin{bmatrix} * & 0 & - \\ 0 & + & - \\ 0 & 0 & - \end{bmatrix} x + \begin{bmatrix} * & + & + \\ + & * & + \\ + & + & * \end{bmatrix}$$

$$\mathcal{C}(\widehat{P})(y) = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} y^2 - \begin{bmatrix} + & 0 & - \\ 0 & + & - \\ 0 & 0 & * \end{bmatrix} y + \begin{bmatrix} + & + & 0 \\ + & + & 0 \\ + & + & 0 \end{bmatrix}$$

More ingredients for a subtraction-(essentially)-free algorithm:
- Choosing parameters: $\alpha = 0$, β from problem magnitudes

Triplets for CR — simpler (stochastic) case

$$\text{triplet}(\hat{B}_k) = (\text{offdiag}(\hat{B}_k), \underline{\mathbf{1}}, C_kB_kA_0\underline{\mathbf{1}})$$

- Don't compute solution $R = C_0\hat{B}_k^{-1}$ to the matrix equation, but invariant pair $U = \hat{B}_k, X = \hat{B}_k^{-1}R\hat{B}_k$

References

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- G Latouche, GT Nguyen The morphing of fluid queues into Markov-modulated Brownian motion arXiv:1311.3359
- JG Xue, SF Xu, RC Li Accurate solutions of M-matrix algebraic Riccati equations. Numer Math 2012
- GT Nguyen, F Poloni Componentwise accurate fluid queue computations using doubling algorithms In preparation