Exploiting displacement structure in the solution of a class of nonsymmetric algebraic Riccati equations

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2nd intl. conference on matrix methods and operator equations
Moscow, 23 July 2007

Outline

Algebraic Riccati equations

Nonsymmetric algebraic Riccati equation (NARE)

$$\begin{aligned} XCX - AX - XE + B = 0 \\ A, B, C, E, X \in \mathbb{R}^{n\times n} \end{aligned} \tag{NARE}$$

Recent interest in the literature e.g. [Guo–Laub '00, Lu '05, Guo–Higham '05, Bini–Iannazzo–Latouche–Meini '06]

Algebraic Riccati equations

Nonsymmetric algebraic Riccati equation (NARE)

$$XCX - AX - XE + B = 0 \qquad \text{(NARE)}$$
$$A, B, C, E, X \in \mathbb{R}^{n\times n}$$

Recent interest in the literature e.g. [Guo–Laub '00, Lu '05, Guo–Higham '05, Bini–Iannazzo–Latouche–Meini '06]

$$X \text{ solves (NARE)} \quad \Leftrightarrow \quad \begin{bmatrix} E & -C \\ B & -A \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (E - CX)$$

$$\text{Solutions} \quad \Leftrightarrow \quad \text{invariant subspaces of } \mathcal{H} := \begin{bmatrix} E & -C \\ B & -A \end{bmatrix}$$

- Explicit calculation of the eigenvectors: numerical problems
- Iterative methods: cost $O(n^3)$/step, quadratic convergence

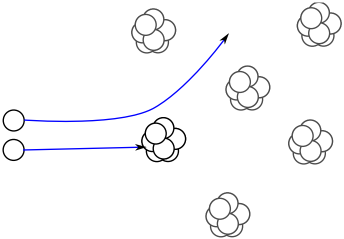
One-group neutron transport equation

$$\left\{(\mu+\alpha)\frac{\partial}{\partial x}+1\right\}\varphi(x,\mu)=\frac{c}{2}\int_{-1}^{1}\varphi(x,\omega)d\omega$$

$$\varphi(0,\mu)=f(\mu),\quad \mu>-\alpha,\quad |\mu|\leqslant 1,$$

$$\lim_{x\to\infty}\varphi(x,\mu)=0.$$

Propagation of neutrons through a slab of shielding material



One-group neutron transport equation

$$\left\{(\mu+\alpha)\frac{\partial}{\partial x}+1\right\}\varphi(x,\mu)=\frac{c}{2}\int_{-1}^{1}\varphi(x,\omega)d\omega$$

$$\Downarrow$$

Reduction to kernel + Gaussian quadrature $\int_0^1 f(x)dx \approx \sum w_i f(x_i)$

$$\Downarrow$$

The resulting equation

$$\Delta X + XD = (Xq+e)(e^T + q^T X) \qquad \text{(NT)}$$

D, Δ "positive" diagonals, $e, q > 0$ vectors

(NT) is a NARE with rank structure:

$$A = \Delta - eq^T,\ B = ee^T,\ C = qq^T,\ E = D - qe^T$$

Algorithms for NARE

1. Newton's method [C.H. Guo–Laub '00]
2. Newton applied to Lu's iteration [Lu '05] — only for (NT)
3. Structured doubling algorithm [X.X. Guo–Lin–Xu '06]
4. Cyclic reduction [Ramaswami '99 and others]

All with cost $O(n^3)$/step, quadratic convergence

Our results

- Structured versions for (NT), with cost $O(n^2)$/step
- Shift technique [He–Meini–Rhee '01] in the structured algorithms
- Interesting connections: $1 = 2$, $3 \subseteq 4$
- New variants to 4

Riccati equations and M-matrices

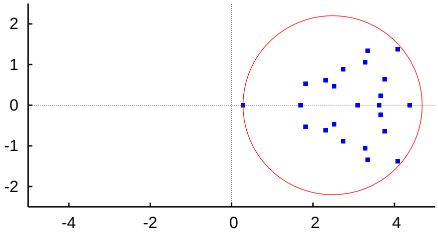
Classical hypothesis — includes case (NT)

$\mathcal{M} = \begin{bmatrix} E & -C \\ -B & A \end{bmatrix}$ is an M-matrix

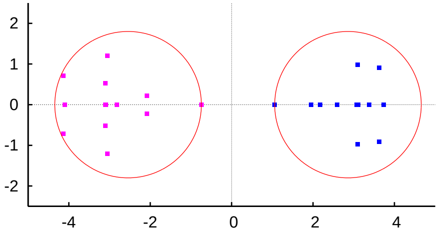
With this assumption,

- $\mathcal{H} = \begin{bmatrix} E & -C \\ B & -A \end{bmatrix}$ has:
  n eigenvalues in the positive half-plane $\Re(\lambda) > 0$
  n in the negative half-plane
  (eventually some on the border)
- Exists S minimal nonnegative solution
- $S \Leftrightarrow$ eigenvalues with $\Re(\lambda) > 0$
- The classical algorithms converge to S

Example



Eigenvalues of $\mathcal{M}$



Eigenvalues of $\mathcal{H}$

Cauchy-like matrices

Displacement operator

$$\nabla_{R,S}(X) := RX - XS$$

R, S diagonal matrices

Cauchy-like matrices: $\nabla_{R,S}(X)$ is low rank $\Leftrightarrow$

$$X_{ij} = \frac{u_i \cdot v_j}{R_{ii} - S_{jj}} \quad \text{whenever } R_{ii} \neq S_{jj}$$

u_i, v_j (*generators*) are $1 \times r$, $r \times 1$ vectors

Usually one requires $R_{ii} \neq S_{jj}$ for all i,j

Instead, we will also need the case $R = S$ (*Trummer-like*):
nothing is known about the main diagonal of X

The GKO algorithm

Solving linear systems with Cauchy-like matrices: GKO algorithm [Gohberg–Kailath–Olshevsky '95]

During Gaussian elimination,

$$M \longrightarrow \begin{bmatrix} c_{11} & c_{12} \\ 0 & C \end{bmatrix} \text{ with } C \text{ Cauchy-like}$$

Instead of updating the elements of C (cost: $O(n^3)$), update its generators (cost: $O(n^2)$)

Trummer-like case is similar:

- Update the diagonal of C as in the traditional Gaussian elimination $O(n^2)$
- Update the other elements as in GKO $O(n^2)$

Lu's algorithm

The resulting equation

$$\Delta X + XD = (Xq + e)(e^T + q^T X) \qquad \text{(NT)}$$

Let $u := Xq + e$, $v^T := e^T + q^T X$

$$(NT) \iff \nabla_{\Delta,-D}(X) = uv^T$$

X is Cauchy-like, and

$$\begin{cases} u = \nabla^{-1}_{\Delta,-D}(uv^T)q + e \\ v = \left(\nabla^{-1}_{\Delta,-D}(uv^T)\right)^T q + e \end{cases} \qquad \text{(LU)}$$

Let $w := \begin{bmatrix} u \\ v \end{bmatrix}$; (LU) is $F(w) = 0$, solve with Newton's method

$$w_{k+1} = w_k - (\nabla F_{w_k})^{-1} F(w_k)$$

The same in $O(n^2)$: ∇F is Trummer-like, use GKO

Newton's algorithm

NARE

$$XCX - AX - XE + B = 0$$

Newton's method applied directly to $R(X) = XCX - AX - XE + B$
The Jacobian is

$$\nabla R_X = I \otimes (A - XC) + (E - CX)^T \otimes I \qquad (n^2 \times n^2 \text{ matrix})$$

Or rather,

$$\nabla R_X(Y) = (A - XC)Y + Y(E - XC) \qquad \text{(SYL)}$$

We need ∇R_X^{-1}: solve (SYL), costs $O(n^3)$ (but slow)

Structured Newton for (NT)

$$\nabla R_X = (E - CX)^T \otimes I_n + I_n \otimes (A - XC) =$$

$$\underbrace{(D^T \otimes I_n + I_n \otimes \Delta)}_{\text{diagonal } n^2 \times n^2} - \underbrace{\begin{bmatrix}(e + X^T q \otimes I_n) & I_n \otimes (e + Xq)\end{bmatrix}}_{n^2 \times 2n} \underbrace{\begin{bmatrix} q^T \otimes I_n \\ I_n \otimes q^T \end{bmatrix}}_{2n \times n^2}$$

Sherman–Morrison–Woodbury formula

$$(\mathcal{D} - UV)^{-1} = \mathcal{D}^{-1} + \mathcal{D}^{-1} U (I_{2n} - V \mathcal{D}^{-1} U)^{-1} V \mathcal{D}^{-1}$$

We reduce to the inversion of $\mathcal{R} = I_{2n} - V\mathcal{D}^{-1}U$, $2n \times 2n$.

Moreover,

- $\mathcal{R}$ is Trummer-like (we can use GKO)
- $\mathcal{R}$ is well-conditioned

Relations between Newton and Lu

Lu: Invert ∇F_{w_k} Newton: Invert $\mathcal{R} = I_{2n} - V_k \mathcal{D}^{-1} U$

∇F_{w_k} and $\mathcal{R}$ have the same structure. A deeper connection?

Theorem

Let u_k, v_k be the iterates of Lu, starting from $u_{-1} = v_{-1} = 0$, and X_k be the iterates of Newton, starting from $X_0 = 0$. Then,

$$\begin{cases} u_k = X_k q + e \\ v_k = X_k^T q + e \end{cases} \qquad \forall k \geqslant 0$$

Interpretation Newton iterates are Cauchy-like:

$$\nabla_{\Delta,-D} X^{(k+1)} = u_{k+1} v_{k+1}^T - (u_{k+1} - u_k)(v_{k+1}^T - v_k^T)$$

Lu performs Newton's iteration working on the generators.

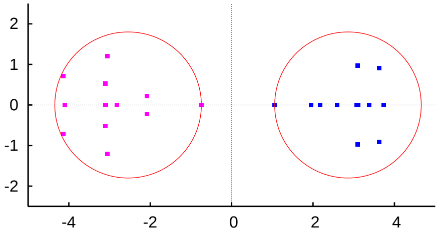
Structured doubling algorithm (SDA)

Structured doubling algorithm

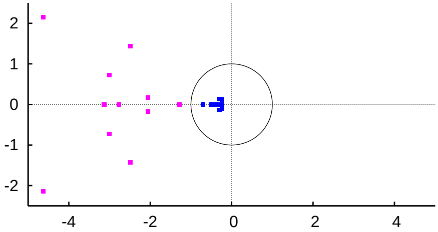
$$
\begin{aligned}
E_{k+1} &= E_k(I - G_k H_k)^{-1} E_k,\\
F_{k+1} &= F_k(I - H_k G_k)^{-1} F_k,\\
G_{k+1} &= G_k + E_k(I - G_k H_k)^{-1} G_k F_k,\\
H_{k+1} &= H_k + F_k(I - H_k G_k)^{-1} H_k E_k,
\end{aligned}
\tag{SDA}
$$

1. Spectral transformation:
   $\mathcal{H} = \begin{bmatrix} E & -C \\ B & -A \end{bmatrix} \mapsto \mathcal{H}_\gamma := (\mathcal{H} + \gamma I)^{-1}(\mathcal{H} - \gamma I)$
2. Block *UL* factorization: $\mathcal{H}_\gamma = \mathcal{U}_0^{-1}\mathcal{L}_0$ con
   $$\mathcal{U} = \begin{bmatrix} I & -G_0 \\ 0 & F_0 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} E_0 & 0 \\ -H_0 & I \end{bmatrix}$$
3. Implicit update $\mathcal{H}_\gamma^{2^k} = \mathcal{U}_k^{-1}\mathcal{L}_k$

Example



Eigenvalues of $\mathcal{H}$



Eigenvalues of $\mathcal{H}_\gamma = (\mathcal{H} + \gamma I)^{-1}(\mathcal{H} - \gamma I)$

Fast SDA for (NT)

$$\mathcal{H}_\gamma^{2^k} = \begin{bmatrix} I & -G_k \\ 0 & F_k \end{bmatrix}^{-1} \begin{bmatrix} E_k & 0 \\ -H_k & I \end{bmatrix}$$

Cauchy-like structure

$$\begin{aligned}
DE_k - E_k D &= (q + G_k e)e^T E_k - E_k q(e^T + q^T H_k),\\
\Delta F_k - F_k \Delta &= (H_k q + e)q^T F_k - F_k e(e^T + q^T G_k),\\
DG_k + G_k \Delta &= (q + G_k e)(e^T + q^T G_k) - E_k q q^T F_k,\\
\Delta H_k + H_k D &= (H_k q + e)(e^T + q^T H_k) - E_k q q^T F_k,
\end{aligned}$$

Instead of updating E_k, F_k, G_k, H_k,

- update the above generators
- store and update the main diagonals of E_k, F_k

Cyclic reduction

NARE $\Leftrightarrow$ eigenvalue problem $\begin{bmatrix} E & -C \\ B & -A \end{bmatrix} u = \lambda u$

Multiply the second block column by λ:

$$\left(\begin{bmatrix} E & 0 \\ B & 0 \end{bmatrix} + \begin{bmatrix} -I & -C \\ 0 & -A \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \lambda^2 \right) u = 0$$

yields a quadratic eigenvalue problem

Theorem

S solves the NARE $\Leftrightarrow$ $\begin{bmatrix} E - CS & 0 \\ S & 0 \end{bmatrix}$ *solves the unilateral equation*

$$\begin{bmatrix} E & 0 \\ B & 0 \end{bmatrix} + \begin{bmatrix} -I & -C \\ 0 & -A \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} X^2 = 0 \qquad \text{(UNI)}$$

Cyclic reduction – the classical algorithm

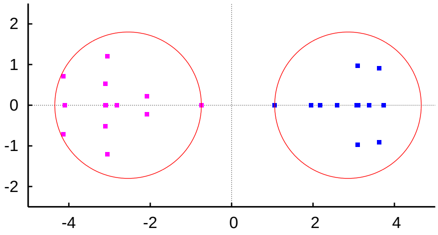
Cyclic reduction [Buzbee–Golub–Nielson, '69]

$$
\begin{aligned}
\mathcal{A}_0^{(k+1)} &= \mathcal{A}_0^{(k)} - \mathcal{A}_{-1}^{(k)}\mathcal{K}^{(k)}\mathcal{A}_1^{(k)} - \mathcal{A}_1^{(k)}\mathcal{K}^{(k)}\mathcal{A}_{-1}^{(k)}, \quad \mathcal{K}^{(k)} = \left(\mathcal{A}_0^{(k)}\right)^{-1}, \\
\mathcal{A}_{-1}^{(k+1)} &= -\mathcal{A}_{-1}^{(k)}\mathcal{K}^{(k)}\mathcal{A}_{-1}^{(k)}, \\
\mathcal{A}_1^{(k+1)} &= -\mathcal{A}_1^{(k)}\mathcal{K}^{(k)}\mathcal{A}_1^{(k)}, \\
\widehat{\mathcal{A}}_0^{(k+1)} &= \widehat{\mathcal{A}}_0^{(k)} - \mathcal{A}_1^{(k)}\mathcal{K}^{(k)}\mathcal{A}_{-1}^{(k)}.
\end{aligned}
\tag{CR}
$$

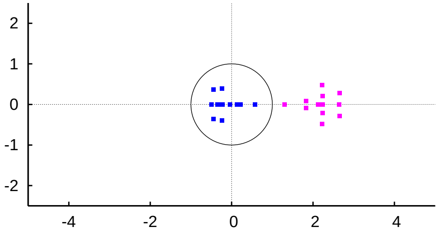
With some assumptions, (CR) converges to the solution of $\mathcal{A}_{-1} + \mathcal{A}_0 X + \mathcal{A}_1 X^2 = 0$ with smaller eigenvalues (in modulus)

1. spectral transformation $\mathcal{H} \mapsto I - t\mathcal{H}$ (shrink-and-shift)
2. apply (CR) to (UNI)

Example



Eigenvalues of $\mathcal{H}$



Eigenvalues of $I - t\mathcal{H}$

Fast cyclic reduction for (NT)

Cauchy-like structure

$$\nabla_{\mathcal{D},\mathcal{D}}\mathcal{A}_{-1}^{(k)} = \mathcal{A}_{-1}^{(k)}\begin{bmatrix} q \\ 0 \end{bmatrix} s_0^{(k)} + \mathcal{A}_0^{(k)}\begin{bmatrix} 0 \\ e \end{bmatrix} t_{-1}^{(k)} + u_0\left[e^T,\ -q^T\right]\mathcal{A}_{-1}^{(k)},$$

$$\nabla_{\mathcal{D},\mathcal{D}}\mathcal{A}_{0}^{(k)} = \mathcal{A}_{-1}^{(k)}\begin{bmatrix} q \\ 0 \end{bmatrix} s_1^{(k)} + \mathcal{A}_0^{(k)}\begin{bmatrix} q \\ 0 \end{bmatrix} s_0^{(k)} + \mathcal{A}_0^{(k)}\begin{bmatrix} 0 \\ e \end{bmatrix} t_0^{(k)} + \mathcal{A}_1^{(k)}\begin{bmatrix} 0 \\ e \end{bmatrix} t_{-1}^{(k)} + u_0\left[e^T,\ -q^T\right]\mathcal{A}_0^{(k)},$$

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Instead of updating $\mathcal{A}_{-1}^{(k)}$, $\mathcal{A}_0^{(k)}$, $\mathcal{A}_1^{(k)}$,

- update the above generators
- store and update the main diagonals of $\mathcal{A}_{-1}^{(k)}$, $\mathcal{A}_1^{(k)}$

Relations between SDA and CR

Theorem

SDA is CR applied to a different reduction (not only for (NT)!)

1. Spectral transformation

$$\mathcal{H} \mapsto \mathcal{H}_\gamma := (\mathcal{H} + \gamma I)^{-1}(\mathcal{H} - \gamma I) = \begin{bmatrix} I & -G_0 \\ 0 & F_0 \end{bmatrix}^{-1} \begin{bmatrix} E_0 & 0 \\ -H_0 & I \end{bmatrix}$$

2. Reduction to a quadratic eigenvalue problem

$$\begin{bmatrix} E_0 & 0 \\ -H_0 & I \end{bmatrix} u = \lambda \begin{bmatrix} I & -G_0 \\ 0 & F_0 \end{bmatrix} u$$

multiply the second block row by λ

$$\left(\begin{bmatrix} E_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -I & G_0 \\ H_0 & -I \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 \\ 0 & F_0 \end{bmatrix} \lambda^2 \right) u = 0 \qquad \text{(SDA-U)}$$

3. (CR) on the unilateral equation associated to (SDA-U)

Much freedom, plenty of room for improvements

A new algorithm

Only when X is a square matrix — (NT) is ok

Idea: in the reduction step, try to make $\mathcal{H}$ triangular

1. shrink-and-shift $\mathcal{H} \to I - t\mathcal{H}$
2. conjugate to make the $(1,2)$ block nonsingular

$$\mathcal{H} \to \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}^{-1} \mathcal{H} \begin{bmatrix} I & M \\ 0 & I \end{bmatrix} = \begin{bmatrix} * & -R(M) \\ * & * \end{bmatrix}$$

3. conjugate again to eliminate the $(1,1)$ block

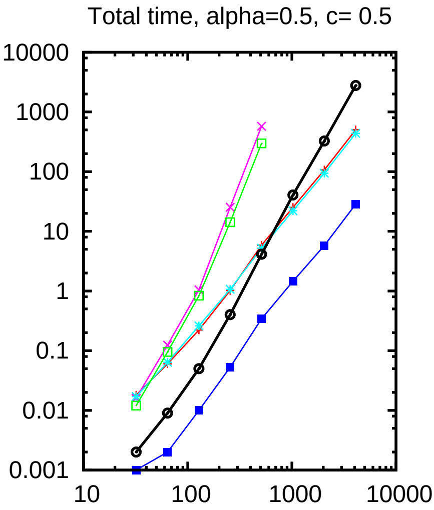
$$\mathcal{H} \to \begin{bmatrix} I & 0 \\ C^{-1}D & I \end{bmatrix}^{-1} \mathcal{H} \begin{bmatrix} I & 0 \\ C^{-1}D & I \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$$

4. A change of variables yields a $n \times n$ unilateral equation $\Rightarrow$ CR.

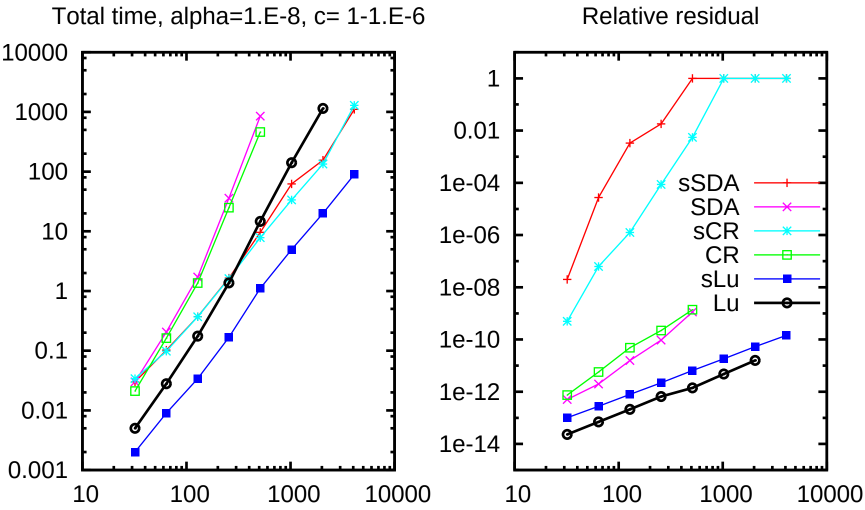
Cheaper to solve, $\frac{38}{3}n^3$/step instead of $\frac{64}{3}n^3$ (SDA)

Similar approach in [Bini–Iannazzo, '03]

Numerical results – noncritical case



Numerical results – quasi-critical case



Results and research lines

- **sLu** is the faster algorithm for (NT)
- **sSDA** and **sCR** could be useful for diagonal + rank r
- Better understanding of the algorithms, unified proofs
- Meaningful results not only for (NT), but for any NARE
- Ideas for new algorithms

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Thanks for your attention!