A note on the location of polynomial roots

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Abstract

We review some known inclusion results for the roots of a polynomial, and adapt them to a conjecture recently presented by S. A. Vavasis. In particular, we provide strict upper and lower bounds to the distance of the closest root of a polynomial $p(z)$ from a given $\zeta \in \mathbb{C}$ such that $p'(\zeta) = 0$.

1 Introduction

Recently S.A. Vavasis [2] has presented the following conjecture.

Conjecture *There exist two universal constants $0 < \iota_1 \leq 1 \leq \iota_2$ with the following property. Let $\xi_1, \dots, \xi_n$ be the roots of a degree-n univariate polynomial $p(z)$. Let $\zeta_1, \dots, \zeta_{n-1}$ be the roots of its derivative. Define*

$$\rho_j = \min_{k=2,\dots,n} \left| \frac{k! p(\zeta_j)}{p^{(k)}(\zeta_j)} \right|^{1/k}, \quad j = 1, \dots, n-1 \tag{1}$$

and the annuli

$$A_j = \{z : \iota_1 \rho_j \leq |z - \zeta_j| \leq \iota_2 \rho_j\}, \quad j = 1, \dots, n-1.$$

Then for each $i = 1, \dots, n$

$$\xi_i \in A_1 \cup \cdots \cup A_{n-1}.$$

The author also refers to an unpublished communication by Giusti et Al., where it is shown that ι_1 exists and can be taken $(\sqrt{5}-1)/2$ and where a sequence of n-degree polynomials is given such that $\lim_n |z - \zeta_j|/\rho_j = +\infty$ so that ι_2 does not exist.

In this note we revisit some known general bounds to the roots of a polynomial from [1], in particular Theorem 6.4b on pages 451,452, and Theorem 6.4e on page 454, and adapt them to the conditions of the Vavasis conjecture.

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More specifically, we show that for any polynomial $p(z)$, and for any ζ such that $p'(\zeta)=0$, there exists a root ξ of $p(z)$ satisfying

$$|\xi-\zeta| \le \rho\sqrt{n/2}, \quad \rho = \min_{k=2,\ldots,n} \left(\frac{k!p(\zeta)}{p^{(k)}(\zeta)}\right)^{1/k},$$

and that the bound is sharp since it is attained by a suitable polynomial.

We provide also some sharp lower bound to $|\xi-\zeta|$ under the condition that $p^{(k)}(\zeta)=0$ for $k\in\Omega$, where Ω is a nonempty subset of $\{1,2,\ldots,n-1\}$.

Moreover, we also show that ι_2 does not exist by providing an example of a sequence $\{p_n(z)\}_n$ of polynomials of degree $n+1$ having a common root ξ, where the ratio $|\xi-\zeta_j^{(n)}|/\rho_j^{(n)}$ is independent of j and tends to infinity as $n^{1-\epsilon}$ for any $i=1,\ldots,n$ and for any $0<\epsilon<1$, where $\zeta_j^{(n)}$ are the roots of $p_n'(z)$.

2 Main results

In this section, after providing a counterexample of the Vavasis conjecture, we review some inclusion theorems of [1], which give lower bounds and upper bounds to the distance of the roots of a polynomial from a given complex number ζ.

2.1 Counterexample

Consider the monic polynomial of degree $n+1$

$$p_n(z) = z^{n+1} - (n+1)z.$$

Clearly $z=0$ is one of its roots, and we have $p_n'(z) = (n+1)(z^n-1)$, so that the roots ζ_i of p_n' are the complex n-th roots of the unity. Define

$$\rho^{(n,k)} = \left|\frac{k!p_n(\zeta)}{p_n^{(k)}(\zeta)}\right|^{1/k}, \qquad \rho^{(n)} = \min_{k=2,\ldots,n+1} \rho^{(n,k)},$$

where ζ stands for any n-th root ζ_i of 1, and observe that $p_n(\zeta)=-n\zeta$, $p_n^{(2)}(\zeta) = n(n+1)\zeta^{-1}$. Therefore, for $k=2$ one has

$$\rho^{(n)} \le \rho^{(n,2)} = \left|\frac{2!p_n(\zeta)}{p_n^{(2)}(\zeta)}\right|^{1/2} = \left|\frac{2n}{n(n+1)}\right|^{1/2} = \sqrt{\frac{2}{n+1}}$$

hence $\rho_n \to 0$ as $n\to\infty$. Observe that this bound is independent of the root ζ_i. The annuli A_i have their centers on the unit circle and for ι_2 constant, their external radii tend to 0 as $n\to\infty$. Thus, for sufficiently large values of n they cannot contain the origin, and this contradicts the conjecture as $z=0$ is a common root to all the polynomials $p_n(z)$.

Moreover, for $z=0$ one has

$$\frac{|z-\zeta_i|}{\rho^{(n)}} \ge \frac{|z-\zeta_i|}{\rho^{(n,2)}} = (n+1)^{1/(n+1)}\sqrt{\frac{n+1}{2}} \ge \sqrt{\frac{n+1}{2}}.$$

That is, the ratio $\frac{|z-\zeta_i|}{\rho^{(n)}}$ can grow as much as $\sqrt{n/2}$. For general k one can easily get

$$\frac{|z-\zeta|}{\rho^{(n,k)}} = \left[\frac{1}{n}\binom{n+1}{k}\right]^{1/k}(n+1)^{\frac{1}{n+1}} \geq \left[\frac{1}{n}\binom{n+1}{k}\right]^{1/k}. \qquad (2)$$

Thus, for a fixed k the ratio $|z-\zeta|/\rho^{(n,k)}$ can grow as much as $n^{1-\frac{1}{k}}$.

2.2 Lower bounds

Let us recall the following result (see [1], Theorem 6.4b).

Theorem 1 *Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a monic polynomial of degree n and ζ any complex number. Assume $a_0 \neq 0$. Then any root ξ of $p(x)$ is such that*

$$\gamma\rho < |\xi-\zeta|, \quad \rho = \rho(\zeta) = \min_{k=2,\ldots,n}\left|k!\frac{p(\zeta)}{p^{(k)}(\zeta)}\right|^{1/k} \qquad (3)$$

where $\gamma = 1/2$.

The following proof of the above theorem can be easily adjusted to the case where ζ is a (numerical) root of some derivative of $p(z)$.

Without loss of generality we may assume $\zeta = 0$. In fact, if $\zeta \neq 0$ consider $\widehat{p}(z) = p(z-\zeta)$ so that $\widehat{p}'(z) = p'(z-\zeta)$ and $\widehat{\rho}(0) = \rho(\zeta)$, and reduce the case to $\zeta = 0$.

From the definition of ρ one has

$$\rho^k \leq k!\left|\frac{p(0)}{p^{(k)}(0)}\right| = \left|\frac{a_0}{a_k}\right|. \qquad (4)$$

Then taking the moduli in both sides of the equation $-a_0 = a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n$ yields

$$1 \leq \sum_{i=1}^{n}\left|\frac{a_i}{a_0}\xi^i\right|$$

which, in view of (4) provides the bound

$$1 \leq \sum_{i=1}^{n} t^i, \quad t = \frac{|\xi|}{\rho},$$

whence

$$1 \leq \frac{t-t^{n+1}}{1-t}.$$

If $t < 1$ then we have $1-t \leq t-t^{n+1} < t$ which implies $t > 1/2$. This proves the bound $|\xi| > \frac{1}{2}\rho$ for any root ξ of $p(z)$.

Observe that the bound is strict since the polynomial $p_n(z) = \sum_{i=1}^{n} z^i - 1$ has a root in the interval $(1/2, 1/2(1+1/n))$ for $n \geq 2$.

The proof of Theorem 1 can be adjusted to the case where ζ satisfies some additional condition. We have the following result:

Proposition 1 *Assume that ζ satisfies the following condition*

$$\theta^i \left| \frac{p^{(i)}(\zeta)}{i! p(\zeta)} \right| \le \epsilon, \quad i \in \Omega = \{i_1, \ldots, i_h\} \subset \{1, 2, \ldots, n-1\}$$

where $0 \le \epsilon < 1/h$, $1 \le h < n$ and θ is an upper bound to $|\zeta - \xi_i|$ for $i = 1, \ldots, n$. Then (3) holds where γ is the only solution in $(1/2, 1)$ of the equation

$$(t-1) \sum_{i \in \Omega} t^i + 2t - 1 + (1-t)h\epsilon = 0. \tag{5}$$

Proof. By following the same arguments of the proof of Theorem 1 with $\zeta = 0$ one obtains

$$1 \le \sum_{i=1}^{n} \left| \frac{a_i}{a_0} \xi^i \right| \le \sum_{i=1,n;\ i \notin \Omega} \left| \frac{a_i}{a_0} \xi^i \right| + h\epsilon \le \sum_{i=1,n;\ i \notin \Omega} t^i + h\epsilon.$$

If $t < 1$, replacing $\sum_{i=1,n;\ i \notin \Omega} t^i = (t - t^{n+1})/(1-t) - \sum_{i \in \Omega} t^i$ in the latter inequality yields $1 - t \le t - t^{n+1} - (1-t) \sum_{i \in \Omega} t^i + (1-t)h\epsilon \le t + (t-1) \sum_{i \in \Omega} t^i + (1-t)h\epsilon$. Whence, $t > \gamma$ where γ is the only solution of (5) in $(1/2, 1)$. $\square$

Let us look at some specific instances of the above result. For $\epsilon = 0$ the condition of the proposition turns into $p^{(i)}(\zeta) = 0$ for $i \in \Omega$. If in addition $\Omega = \{1\}$ one finds the condition $p'(\zeta) = 0$ of the Vavasis conjecture and (5) turns into $t^2 + t - 1 = 0$ that implies $\gamma = (\sqrt{5} - 1)/2 = 0.618...$. Weaker bounds are obtained assuming $\epsilon = 0$ and $\Omega = \{k\}$ for some $k > 1$ since the only root of the polynomial $t^{k+1} - t^k + 2t - 1$ in $(1/2, 1)$ is lower than $(\sqrt{5} - 1)/2$.

Better bounds are obtained if ζ is a root of multiplicity h of $p'(z)$; in fact, γ is the only positive root of the polynomial $t^{h+1} + t - 1$. In particular, if $h = 2$ then $\gamma = 0.682...$, if $h = 3$, $\gamma = 0.724...$.

If ζ is close to a root of $p'(z)$, so that the condition $\theta |p'(\zeta)/p(\zeta)| < \epsilon$ for some "small" ϵ is satisfied, then $\gamma = (\sqrt{5} - 1)/2 - \epsilon(1 + 3/\sqrt{5}) + O(\epsilon^2)$.

For $\epsilon = 0$ the bound in the above proposition is strict since it is asymptotically attained by the polynomial $t^n - (t-1) \sum_{i \in \Omega} t^i - 2t + 1$. The advantage of this bound is that it allows to compute sharper values for γ just by solving a low degree equation if Ω is made up by small integers.

Slightly better lower bounds can be obtained from the following known result of [1] which requires to compute a positive root of a polynomial of degree n.

Theorem 2 *Any root ξ of $p(z)$ is such that $|\xi| \ge \sigma$, where σ is the only positive solution to the equation $|a_0| = \sum_{i=1}^{n} t^i |a_i|$.*

2.3 Upper bounds

Throughout this section we denote

$$\rho^{(k)} = \left(k! p(\zeta)/p^{(k)}(\zeta) \right)^{1/k}, \quad \rho = \min_k \rho^{(k)}$$

for a given $\zeta \in \mathbb{C}$. Concerning upper bounds to the distance of a root from ζ we recall the following result of [1] (Theorem 6.4e, page 454).

Theorem 3 *For any $\zeta \in \mathbb{C}$ there exists a root ξ of $p(z)$ such that*

$$|\xi - \zeta| \le \rho^{(k)} \binom{n}{k}^{1/k}, \quad k = 1, \ldots, n. \tag{6}$$

Observe that, for $k = 2$ one has

$$|\xi - \zeta| \le \rho^{(2)} \sqrt{n(n-1)/2}, \tag{7}$$

while

$$|\xi - \zeta| \le \min_k \binom{n}{k}^{1/k} \rho^{(k)} \le \max_k \binom{n}{k}^{1/k} \rho \le n\rho. \tag{8}$$

The bound (8) is sharp since it is attained by the polynomial $p(z) = (z - n)^n$ with $\zeta = 0$. In fact, it holds $\rho = \rho^{(1)} = 1$ and $p(z)$ has roots of modulus n.

Under the condition $p'(\zeta) = 0$ the bounds (6), (7) and (8) can be substantially improved. In fact we may prove the following result

Proposition 2 *For any $\zeta \in \mathbb{C}$ such that $p'(\zeta) = 0$ there exists a root ξ of $p(z)$ such that*

$$|\xi - \zeta| \le \begin{cases} \rho^{(2)} \sqrt{n/2} \\ \rho^{(3)} \sqrt[3]{n/3} \\ \rho^{(k)} \sqrt{n} \left(\frac{1}{k} \prod_{i=2}^{\lfloor k/2 \rfloor} \left(\frac{1}{n} + \frac{1}{2i-1} + \frac{1}{2i-2} \right) \right)^{1/k} \quad \textit{for} \;\; 4 \le k \le n \end{cases} \tag{9}$$

Moreover,

$$|\xi - \zeta| \le \rho \sqrt{\frac{n}{2}} \tag{10}$$

Proof. Without loss of generality we may assume $\zeta = 0$ and $a_0 = 1$ so that the polynomial can be written as $p(z) = 1 + a_2 z^2 + \ldots + a_n z^n$. Recall the Newton identities [1], page 455:

$$k a_k = -s_k - \sum_{i=1}^{k-1} a_i s_{k-i}, \quad k = 1, 2, \ldots,$$

where $s_k = \sum_{i=1}^{n} \xi_i^{-k}$ are the power sums of the reciprocal of the roots ξ of $p(z)$. Clearly, $a_1 = s_1 = 0$ so that for $k \ge 4$ the Newton identities turn into

$$k a_k = -s_k - \sum_{i=2}^{k-2} a_i s_{k-i}, \quad k = 4, 5, \ldots. \tag{11}$$

Let $\Delta = \min_i |\xi_i|$ so that $|s_k| \leq n\Delta^{-k}$. It holds $|2a_2| = |s_2| \leq n\Delta^{-2}$, $|3a_3| = |s_3| \leq n\Delta^{-3}$ and

$$k|a_k| \leq \Delta^{-k} n(1 + \sum_{i=2}^{k-2} |a_i|\Delta^i), \quad k \geq 4.$$

Denoting $\gamma_k = n(1 + \sum_{i=2}^{k-2} |a_i|\Delta^i)$, for $k \geq 4$ and $\gamma_2 = \gamma_3 = n$, by using the induction argument one easily finds that

$$\begin{array}{l} k|a_k| \leq \Delta^{-k}\gamma_k \\ \gamma_k \leq \gamma_{k-1} + \frac{n}{k-2}\gamma_{k-2}, \quad k \geq 4 \\ \gamma_2 = \gamma_3 = n. \end{array} \tag{12}$$

The above expression provides the bound

$$\Delta \leq \rho^{(k)} \left(\frac{\gamma_k}{k}\right)^{1/k} \tag{13}$$

so that it remains to give upper bounds to γ_k. Since $\gamma_2 = \gamma_3 = n$, from (13) we deduce (9) for $k = 2, 3$. For the general case $k \geq 4$, we express the recurrence (12) in matrix form as

$$\begin{bmatrix} \gamma_{k+1} \\ \gamma_k \end{bmatrix} \leq \begin{bmatrix} 1 & \frac{n}{k-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_k \\ \gamma_{k-1} \end{bmatrix},$$

where the inequality holds component-wise. Applying twice the above bound yields

$$\begin{bmatrix} \gamma_{k+1} \\ \gamma_k \end{bmatrix} \leq \begin{bmatrix} 1 + \frac{n}{k-1} & \frac{n}{k-2} \\ 1 & \frac{n}{k-2} \end{bmatrix} \begin{bmatrix} \gamma_{k-1} \\ \gamma_{k-2} \end{bmatrix}. \tag{14}$$

Whence, since $\gamma_2 = \gamma_3 = n$, one finds that γ_{2i} and γ_{2i+1} are polynomials in n of degree i. Denoting

$$\gamma_{2i} = n^i \delta_{2i}, \qquad \gamma_{2i+1} = n^i \delta_{2i+1}, \tag{15}$$

we may give upper bounds to δ_k. In fact, from (14) with $k = 2i$ it holds

$$\begin{bmatrix} \delta_{k+1} \\ \delta_k \end{bmatrix} \leq \begin{bmatrix} \frac{1}{n} + \frac{1}{k-1} & \frac{1}{k-2} \\ \frac{1}{n} & \frac{1}{k-2} \end{bmatrix} \begin{bmatrix} \delta_{k-1} \\ \delta_{k-2} \end{bmatrix}. \tag{16}$$

Let us denote W_k the matrix in the right-hand side of (16), so that for $n \geq 4$ we have

$$\begin{bmatrix} \delta_{2i+1} \\ \delta_{2i} \end{bmatrix} = W_{2i} W_{2(i-1)} \cdots W_4 \begin{bmatrix} \delta_3 \\ \delta_2 \end{bmatrix}. \tag{17}$$

Since for $n \geq 4$ we have $||W_k||_\infty = \frac{1}{n} + \frac{1}{k-1} + \frac{1}{k-2}$, taking norms in (17) yields

$$||(\delta_{2i+1}, \delta_{2i})||_\infty \leq \prod_{j=2}^{i} ||W_{2j}||_\infty ||(\delta_3, \delta_2)||_\infty \leq \prod_{j=2}^{i} \left(\frac{1}{n} + \frac{1}{2j-1} + \frac{1}{2j-2}\right),$$

since $||(\delta_3, \delta_2)||_\infty = ||(1,1)||_\infty = 1$. In view of (13) and (15) this proves (9).

In order to prove the bound (10), from (13) it is sufficient to prove that

$$\gamma_k \le k(\sqrt{\frac{n}{2}})^k. \tag{18}$$

We prove the latter bound by induction on k for $2 \le k \le n$. For $k = 2, 3$, the inequality (18) is true since $\gamma_2 = \gamma_3 = n$. Moreover, from (12) one has $\gamma_4 \le \gamma_3 + \frac{n}{2}\gamma_2 = n(n+2)/2$ so that (18) is satisfied also for $k = 4$. Now we assume that the bound (18) is true for k and $k-1$, where $k \ge 4$ and we prove it for $k+1 \le n$, i.e., $\gamma_{k+1} \le (k+1)(\sqrt{n/2})^{k+1}$. From (12) and from the inductive assumption one has $\gamma_{k+1} = \left(\sqrt{\frac{n}{2}}\right)^{k+1}\left(k\sqrt{\frac{2}{n}} + 2\right)$. Therefore it is sufficient to prove that $k\sqrt{\frac{2}{n}} + 2 \le k+1$, that is, $\sqrt{\frac{n}{2}} \ge \frac{k}{k-1}$ which is satisfied for $n \ge k \ge 4$. This completes the proof. □

Observe that the bound of Theorem 2 is sharp since it is attained by the polynomial $p(z) = (z^2 - m)^m$ with $\zeta = 0$, where $n = 2m$. In fact, $p'(0) = 0$, $\rho = \rho^{(2)} = 1$ and the roots of $p(z)$ have moduli $\sqrt{n/2}$.

If ζ is such that $p^{(j)}(\zeta) = 0$, $j = 1, \ldots, h$, then from the Newton identities one finds that $s_i = a_i = 0$, $i = 1, \ldots, h$ so that equation (11) turns into

$$ka_k = -s_k - \sum_{i=h+1}^{k-h-1} a_i s_{k-i}, \quad k \ge 2(h+1).$$

By following the same argument used in the proof of Proposition 2 we can prove that there exists a root ξ of $p(z)$ such that

$$|\xi - \zeta| \le \rho^{(h+i)} \sqrt[h+i]{\frac{n}{h+1}}, \quad i = 1, \ldots, h+1.$$

References

[1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1, Wiley, 1974.

[2] S. A. Vavasis, A conjecture that the roots of a univariate polynomial lie in a union of annuli (Interim Revised Version), arXiv:math.CV/0606194 v3, 28 Jul 2006.