Nonsingular systems of generalized Sylvester equations: an algorithmic approach*

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Abstract

We consider the uniqueness of solution (nonsingularity) of systems of r generalized Sylvester and $\star$-Sylvester equations with $n \times n$ coefficient matrices. After several reductions, we show that it is sufficient to analyze periodic systems having, at most, one generalized $\star$-Sylvester equation. We provide characterizations for the nonsingularity in terms of spectral properties of either matrix pencils or formal matrix products, both constructed from the coefficients of the system. The proposed approach uses the periodic Schur decomposition, and leads to an $O(n^3 r)$ algorithm for computing the (unique) solution. We prove that the proposed algorithm is backward stable. The asymptotic cost and the stability are then verified by some numerical experiments.

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1 Introduction

The *generalized Sylvester* equation

$$AXB - CXD = E, \tag{1}$$

goes back to, at least, the early 20th century [24]. Here the unknown X, the coefficients A, B, C, D, and the right-hand side E are complex matrices of appropriate size. This equation has attracted much attention since the 1970s, mainly due to its appearance in applied problems (see, for instance, [7, 18, 19, 21, 22]).

Another related interesting equation is the generalized $\star$-Sylvester matrix equation

$$AXB - CX^\star D = E, \tag{2}$$

where the unknown X, the coefficients A, B, C, D, and the right-hand side E are again complex matrices of appropriate size, and $\star$ can be either the transpose ($\top$) or the conjugate transpose ($*$) operator. Notice that, if $\star = \top$, then the equation can be seen as a linear system in the entries of the unknown X, while if $\star = *$, the equation is no more linear in the entries of X because of the conjugation. Nevertheless, with the usual isomorphism $\mathbb{C} \cong \mathbb{R}^2$, obtained by splitting the real and imaginary parts, it turns out to be a linear system with respect to the real entries $\mathrm{re}(X)$ and $\mathrm{im}(X)$.

One could argue that, in some sense, solving generalized Sylvester and $\star$-Sylvester equations is an elementary problem both from the theoretical and the computational point of view, since they are equivalent to linear systems. Nevertheless, there has been great interest in giving conditions on the existence and uniqueness of solution based just on properties of certain small-sized matrix pencils constructed from the coefficients. For instance, when all matrix coefficients are square, it is known that (1) has a unique solution if and only if the two pencils $A - \lambda C$ and $D - \lambda B$ have disjoint spectra [7, Th. 1], whereas the uniqueness of solution of (2) depends on spectral properties of the matrix pencil $\left[\begin{smallmatrix} \lambda D^\star & B^\star \\ A & -\lambda C \end{smallmatrix}\right]$ (see [9, Th. 15]).

On the other hand, if all matrix coefficients are square and of size n, then the resulting linear system has size n^2 or $2n^2$. From the computational point of view, solving a linear system of size n^2 with standard (non-structured) algorithms may be prohibitive, since they result in a method which approximates the solution in $O(n^6)$ (floating point) arithmetic operations (flops). However, dealing with the matrix coefficients it is possible to get an algorithm requiring only $O(n^3)$ flops [7].

Recently, systems of coupled generalized Sylvester and $\star$-Sylvester equations have been considered, and useful conditions on the existence of solutions have been derived in [11]. Here, we consider the same kind of systems and provide further characterizations for the uniqueness of their solution, for any right-hand side, based on certain spectral conditions on their matrix coefficients. It is worth to emphasize that, while in [11] non-square coefficients are allowed, as long as the matrix products are well-defined, here we assume that all coefficient matrices, as well as the unknowns, are square of size $n \times n$. This choice has been made because the problem of nonsingularity, even for just one equation, presents certain additional subtleties when the coefficient matrices are not square or they are square with different sizes (see [10]). In the assumption that all matrix coefficients are square and of size n, such a system of matrix equations is equivalent to a square linear system, which has a unique solution, for any right-hand side, if and only if the coefficient matrix is nonsingular. For this reason, we will use the term *nonsingular system* as a synonym of a system having a unique solution (for any right-hand side).

The *systems of generalized Sylvester and $\star$-Sylvester equations* that we consider are of the form

$$A_k X_{\alpha_k}^{s_k} B_k - C_k X_{\beta_k}^{t_k} D_k = E_k, \qquad k = 1, \ldots, r, \tag{3}$$

where all matrices involved are complex and of size $n \times n$, the indices α_i, β_i of the unknowns are positive integers and can be equal or different to each other, and $s_i, t_i \in \{1, \star\}$.

Our approach starts by reducing the problem on the nonsingularity of (3) to the special case of *periodic* systems of the form

$$\left\{ \begin{array}{rcl} A_k X_k B_k - C_k X_{k+1} D_k & = & E_k, \quad k = 1, \ldots, r-1, \\ A_r X_r B_r - C_r X_1^s D_r & = & E_r, \end{array} \right. \tag{4}$$

where $s \in \{1, \star\}$. We will provide an explicit characterization of nonsingularity only for periodic systems like (4). However, our reduction allows one to get a characterization for any system like (3) after undoing all changes that take the system (3) into (4). Since these systems can be seen as linear systems, the criteria for nonsingularity do not depend on the right-hand sides E_k, but only on the coefficient matrices A_k, B_k, C_k, D_k, for $k = 1, \ldots, r$.

Periodic systems of Sylvester equations naturally arise in the context of discrete-time periodic systems, and they have been analyzed by several authors (see, for instance, [1, 12, 13, 23]). Byers and Rhee provided in the unpublished work [5] a characterization for the nonsingularity of (4) with $s = 1$, together with an $O(n^3 r)$ algorithm to compute the solution.

The first contribution of the present work is the reduction of a general system of Sylvester and $\star$-Sylvester equations (3) to several disjoint systems of periodic type (4), where all equations are generalized Sylvester, with the exception of the last one that may be either a generalized Sylvester or a generalized $\star$-Sylvester equation. We note that neither the coefficient matrices, nor the number of equations in the original and the reduced system necessarily coincide.

As a second contribution, we provide a characterization for the nonsingularity of (4) for the cases $s = *, \top$ (i. e., $s = \star$, according to our notation). This characterization appears in two different formulations. The first one is given in terms of the spectrum of *formal products* constructed from the coefficients of the system (the case $s = 1$ is treated in Theorem 4 and the case $s = \star$ in Theorem 5). The second formulation, valid for $s = \star$, is given in terms of spectral properties of a block-partitioned matrix pencil of size $(2rn) \times (2rn)$ constructed in an elementary way from the coefficient matrices (see Theorem 6). This characterization extends the one in [9] for the single equation (2), and it is in the same spirit as the one obtained in [5] for periodic systems with $s = 1$.

The third contribution of the paper is to provide an $O(n^3 r)$ algorithm to compute the unique solution of a nonsingular system. Our algorithm is a Bartels-Stewart like algorithm, based on the periodic Schur form [3]. It extends the one given in [5] for systems of Sylvester equations only, the one provided in [8] for the $\star$-Sylvester equation $AX + X^\star D = E$, and the one outlined in [6, §4.2] for (2).

We note that extending the results of [5] to include $\star$-Sylvester equations is not a trivial endeavour: the presence of transpositions creates additional dependencies between the data, hence we need a different strategy to reduce the coefficient matrices to a triangular form, and the resulting criteria have a significantly different form.

Throughout the manuscript, $\mathfrak{i}$ denotes the imaginary unit, that is, $\mathfrak{i}^2 = -1$. We also denote by $M^{-\star}$ the inverse of the invertible matrix $M^\star$, with $\star$ being $*$ or $\top$. A pencil $\mathcal{Q}(\lambda)$ is said to be *regular* if it is square and $\det \mathcal{Q}(\lambda)$ is not identically zero. We will use the symbol $\Lambda(\mathcal{Q})$ to denote the *spectrum* of a regular matrix pencil $\mathcal{Q}(\lambda)$, that is the set of values λ such that $\mathcal{Q}(\lambda)$ is singular (including ∞ if the degree of $\det \mathcal{Q}(\lambda)$ is smaller than the size of the pencil). For simplicity, we use the term *system of Sylvester-like equations* for a system of generalized Sylvester and $\star$-Sylvester equations.

The paper is organized as follows. In Section 2 the periodic Schur decomposition and the concept of formal matrix product are recalled. Section 3 hosts the main theoretical results of the paper, whose proofs are deferred to Section 6, after Sections 4 and 5, that are devoted to some successive simplifications of the problem which are useful for the proofs. Section 7 is devoted to describe and analyze an efficient algorithm for the solution of systems of Sylvester-like equations. Finally, in Section 8 we draw some conclusions.

2 Periodic Schur decomposition of formal matrix products

In order to state and prove the nonsingularity results for a system of Sylvester-like equations and to design an efficient algorithm to compute the solution, we need to introduce several results and definitions that extend the ideas of matrix pencils and generalized eigenvalues to products of matrices of an arbitrary number of factors. These are standard tools in the literature (see, for instance, [12, 13]).

Theorem 1 (Periodic Schur decomposition [3])**.** *Let M_k, N_k, for $k = 1, \ldots, r$, be two sequences of $n \times n$ complex matrices. Then there exist unitary matrices Q_k, Z_k, for $k = 1, \ldots, r$, such that*

$$Q_k^* M_k Z_k = T_k, \qquad Q_k^* N_k Z_{k+1} = R_k, \qquad k = 1, \ldots, r \tag{5}$$

where T_k, R_k are upper triangular and $Z_{r+1} = Z_1$.

If the matrices N_k are invertible, Theorem 1 means that we can apply suitable unitary changes of bases to the product

$$\Pi = N_r^{-1} M_r N_{r-1}^{-1} M_{r-1} \cdots N_1^{-1} M_1 \tag{6}$$

to make all its factors upper triangular simultaneously. More precisely,

$$Z_1^{-1} \Pi Z_1 = R_r^{-1} T_r R_{r-1}^{-1} T_{r-1} \cdots R_1^{-1} T_1.$$

In this case, the eigenvalues of Π are

$$\lambda_i = \frac{(T_1)_{ii}(T_2)_{ii} \cdots (T_r)_{ii}}{(R_1)_{ii}(R_2)_{ii} \cdots (R_r)_{ii}}, \quad i = 1, 2, \ldots, n. \tag{7}$$

Even when some of the N_k matrices are not invertible, we call the expression (6) a *formal matrix product*, and (5) a *formal periodic Schur form* of the product. If $(T_1)_{ii}(T_2)_{ii} \cdots (T_r)_{ii} = (R_1)_{ii}(R_2)_{ii} \cdots (R_r)_{ii} = 0$, for some $i \in \{1, 2, \ldots, n\}$, we call the formal product *singular*; otherwise, we call it *regular*. If Π is regular, it makes sense to consider the ratios λ_i defined in (7), with the convention that $\frac{a}{0} = \infty$ for $a \neq 0$. We call these ratios the *eigenvalues* of the regular formal matrix product Π. The set of eigenvalues of Π is called, as usual, the *spectrum of* Π, and we denote it by $\Lambda(\Pi)$.

We also define the eigenvalues of a formal matrix product of the form

$$\Pi = M_r N_{r-1}^{-1} M_{r-1} \cdots N_1^{-1} M_1 N_r^{-1}$$

(i. e., one in which the exponent -1 appears in the factors in *even* positions) by the same formula (7).

Remark 2. For the notion of eigenvalues of formal products to be well defined, one should prove that it does not depend on the choice of the (non-unique) decomposition (5). If all N_i matrices are nonsingular, then this is evident because they coincide with the eigenvalues obtained by performing the inversions and computing the actual product Π. If some of the N_i are singular, then we can use a continuity argument to show that the λ_i are the limits, as $\varepsilon \to 0$, of the eigenvalues of

$$(N_r + \varepsilon P_r)^{-1} M_r (N_{r-1} + \varepsilon P_{r-1})^{-1} M_{r-1} \cdots (N_1 + \varepsilon P_1)^{-1} M_1$$

for each choice of the nonsingular matrices $P_1, P_2, \ldots, P_r$ that make the factors $N_k + \varepsilon P_k$ invertible, for all $k = 1, \ldots, r$.

3 Main results

Here we state the characterizations for the nonsingularity of a periodic system ot type (4) for each of the three possible cases $s \in \{1, \top, *\}$ (the proofs will be given in Section 6). Later, in Section 4, we will show that these characterizations are enough to get a characterization of nonsingularity of the general system (3).

We recall the following definition.

Definition 3. (Reciprocal free and $*$-reciprocal free set [4, 20]). Let $\mathcal{S}$ be a subset of $\mathbb{C} \cup \{\infty\}$. We say that $\mathcal{S}$ is

(a) reciprocal free if $\lambda \neq \mu^{-1}$, for all $\lambda, \mu \in \mathcal{S}$;

(b) $*$-reciprocal free if $\lambda \neq (\overline{\mu})^{-1}$, for all $\lambda, \mu \in \mathcal{S}$.

This definition includes the values $\lambda = 0, \infty$, with the customary assumption $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

For brevity, we will refer to a $\star$-*reciprocal free set* to mean either a reciprocal free and a $*$-reciprocal free set.

The characterization comes in two different forms. The first one uses eigenvalues of formal matrix products. More precisely, we have the following results.

Theorem 4. *Let $A_k, B_k, C_k, D_k \in \mathbb{C}^{n\times n}$, for $k = 1, \dots, r$. The system*

$$\left\{\begin{array}{rcl} A_kX_kB_k - C_kX_{k+1}D_k & = & E_k, \quad k = 1, \dots, r-1, \\ A_rX_rB_r - C_rX_1D_r & = & E_r, \end{array}\right.$$

is nonsingular if and only if the two formal matrix products

$$C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1 \quad \text{and} \quad D_rB_r^{-1}D_{r-1}B_{r-1}^{-1}\cdots D_1B_1^{-1} \tag{8}$$

are regular and they have disjoint spectra.

Theorem 5. *Let $A_k, B_k, C_k, D_k \in \mathbb{C}^{n\times n}$, for $k = 1, \dots, r$. The system*

$$\left\{\begin{array}{rcl} A_kX_kB_k - C_kX_{k+1}D_k & = & E_k, \quad k = 1, \dots, r-1, \\ A_rX_rB_r - C_rX_1^\star D_r & = & E_r, \end{array}\right.$$

is nonsingular if and only if the formal matrix product

$$\Pi = D_r^{-\star}B_r^{*}D_{r-1}^{-\star}B_{r-1}^{\star}\cdots D_1^{-\star}B_1^{\star}C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1 \tag{9}$$

is regular and

- *if $\star = *$, then $\Lambda(\Pi)$ is a $*$-reciprocal-free set,*
- *if $\star = \top$, then $\Lambda(\Pi) \setminus \{-1\}$ is a reciprocal-free set, and the multiplicity of $\lambda = -1$ as an eigenvalue of Π is at most 1.*

The second characterization involves eigenvalues of matrix pencils. In what follows, the notation $\mathfrak{R}_p$ stands for the set of pth roots of unity, namely, $\mathfrak{R}_p := \{e^{2\pi \mathrm{i} j/p},\ j = 0, 1, \dots, p-1\}$.

Theorem 6. *Let $A_k, B_k, C_k, D_k \in \mathbb{C}^{n\times n}$, for $k = 1, \ldots, r$. The periodic system* (4)*, with $s = \star$, is nonsingular if and only if the matrix pencil*

$$\mathcal{Q}(\lambda) := \begin{bmatrix} \lambda A_1 & C_1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \lambda A_r & C_r & & & \\ & & & \lambda B_1^\star & D_1^\star & & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & D_{r-1}^\star \\ -D_r^\star & & & & & & \lambda B_r^\star \end{bmatrix} \tag{10}$$

is regular and

(i) *if $\star = *$, then $\Lambda(\mathcal{Q})$ is $*$-reciprocal-free, and*

(ii) *if $\star = \top$, then $\Lambda(\mathcal{Q}) \setminus \mathfrak{R}_{2r}$ is reciprocal free and the multiplicity of ξ, for any $\xi \in \mathfrak{R}_{2r}$, is at most* 1*.*

Theorem 6 is an extension of [9, Th. 15], where the case of a single generalized $\star$-Sylvester equation is treated. It also resembles the characterization obtained in [5, Th. 3] for systems of generalized Sylvester equations (i.e., without $\star$). We reproduce this last result here, for completeness.

Theorem 7 (Byers and Rhee, [5])**.** *The periodic system* (4)*, with $s = 1$, is nonsingular if and only if the matrix pencils*

$$\begin{bmatrix} \lambda A_1 & C_1 & & \\ & \lambda A_2 & \ddots & \\ & & \ddots & C_{r-1} \\ C_r & & & \lambda A_r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda D_1 & B_1 & & \\ & \lambda D_2 & \ddots & \\ & & \ddots & B_{r-1} \\ B_r & & & \lambda D_r \end{bmatrix}$$

are regular and have disjoint spectra.

Our strategy to prove Theorems 4, 5, and 6 for periodic systems (4) relies on several steps. First, we use the fact that the system is equivalent to a system with triangular coefficients, as shown in Section 5.1. Second, in Section 5.2, when $s = 1$ or $s = \top$, we transform the system of matrix equations with triangular coefficients to an equivalent linear system that is block upper triangular in a suitable basis (given by an appropriate order of the unknowns). The remaining case $s = *$ is reduced to the case $s = 1$ in Section 5.3. Third, we prove in Section 6 that the diagonal blocks of the matrix coefficient of the resulting block triangular system are invertible if and only if the conditions in the statement of Theorems 4, 5, and 6 hold.

4 Reducing the problem to periodic systems

In this section, we are going to show how to reduce the problem of nonsingularity of a general system (3) to the question on nonsingularity of periodic systems (4) with at most one $\star$ in the last equation.

4.1 Reduction to an irreducible system

We say that the system (3) of r equations in s unknowns is *reducible* if there are $0 < k < s$ unknowns appearing only in $0 < h < r$ equations and the remaining $s-k$ unknowns appear only in the remaining $r-h$ equations. In other words, a reducible system can be partitioned in two systems with no common unknowns. A system is said to be *irreducible* if it is not reducible.

Let $\mathbb{S}$ be a system of r ordered equations like (3). Let $\{1,\dots,r\} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\ell$ be a partition of the set of indices. Then we denote by $\mathbb{S}(\mathcal{I}_j)$, for $j = 1,\dots,\ell$, the system of equations comprising the equations with indices in $\mathcal{I}_j$.

Proposition 8. *Let $\mathbb{S}$ be a system (3) with r equations. There exists a partition $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\ell$ of $\{1,\dots,r\}$ such that, for each $j = 1,\dots,\ell$, the system $\mathbb{S}(\mathcal{I}_j)$ is irreducible.*

Proof. We proceed by induction on r. If $r = 1$ the system has only one equation and thus it is irreducible. Let $r \geq 1$ and consider a system with $r+1$ equations. Either it is irreducible or it can be split in two systems of smaller size and we can apply induction on each of these systems. □

Proposition 8 shows that every system can be split into irreducible systems. To determine if a system is nonsingular, it is sufficient to answer the same question for its irreducible components, as stated in the following result.

Proposition 9. *Let $\mathbb{S}$ be the system (3) with s matrix unknowns, and let $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\ell$ be a partition of $\{1,\dots,r\}$ such that each system $\mathbb{S}(\mathcal{I}_j)$ is irreducible, for $j = 1,\dots,\ell$. The system $\mathbb{S}$ is nonsingular if and only if the system $\mathbb{S}(\mathcal{I}_j)$ is nonsingular, for each $j = 1,\dots,\ell$.*

Proof. We shall show directly that $\mathbb{S}$ has a unique solution if and only if $\mathbb{S}(\mathcal{I}_j)$ has a unique solution for each $j = 1,\dots,\ell$. Any solution of $\mathbb{S}$ yields a solution of $\mathbb{S}(\mathcal{I}_j)$, for each $j = 1,\dots,\ell$, and viceversa. Let us assume that $\mathbb{S}$ has two different solutions $(X_1,\dots,X_s)$ and $(Y_1,\dots,Y_s)$. Then there exists some $1 \leq p \leq s$ such that $X_p \neq Y_p$. If $p \in \mathcal{I}_q$, for some $1 \leq q \leq \ell$, then $\mathbb{S}(\mathcal{I}_q)$ has two different solutions, the first one containing X_p and the second one containing Y_p. Conversely, if not every system $\mathbb{S}(\mathcal{I}_j)$ is nonsingular, then there is some $1 \leq q \leq \ell$ such that either $\mathbb{S}(\mathcal{I}_q)$ is not consistent or it has two different solutions. In the first case, the whole system $\mathbb{S}$ would not be consistent either. If $\mathbb{S}(\mathcal{I}_q)$ has two different solutions, $(X_1,\dots,X_{s_q})$ and $(Y_1,\dots,Y_{s_q})$, and $\mathbb{S}(\mathcal{I}_j)$ is consistent, for any $j \neq q$, then we can construct two different solutions of $\mathbb{S}$ by completing with $(X_1,\dots,X_{s_q})$ and $(Y_1,\dots,Y_{s_q})$, respectively, the solutions of the remaining $\mathbb{S}(\mathcal{I}_j)$ for $j \neq q$. □

Proposition 10. *Let $\mathbb{S}$ be the system* (3) *with r matrix unknowns with size $n \times n$ and let $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\ell$ be a partition of $\{1, \ldots, r\}$ such that each system $\mathbb{S}(\mathcal{I}_j)$ is irreducible, for $j = 1, \ldots, \ell$. Let r_j and s_j be the number of matrix equations and unknowns, respectively, of $\mathbb{S}(\mathcal{I}_j)$. If the system $\mathbb{S}$ has a unique solution then $r_j = s_j$, for $j = 1, \ldots, \ell$.*

Proof. If an irreducible system with $\widehat{r}$ equations and $\widehat{s}$ unknowns has unique solution, then $\widehat{s} \leq \widehat{r}$, since otherwise this system, considered as a linear system on the entries of the matrix unknowns, would have more unknowns than equations.

Now, by contradiction, assume that $r_j \neq s_j$, for some $1 \leq j \leq \ell$. Then, since $\sum_{j=1}^{\ell} r_j = \sum_{j=1}^{\ell} s_j = r$, there exists some $1 \leq p \leq \ell$ such that $r_p < s_p$. Thus the system $\mathbb{S}(\mathcal{I}_p)$ cannot have unique solution, and this contradicts Proposition 9. □

The previous results show that, in order to analyze the nonsingularity of a system of r matrix equations in r matrix unknowns, we may assume that the system is irreducible.

Moreover, Proposition 9 shows that a first step to compute the unique solution of a system of type (3) consists in splitting the system into irreducible systems and solving them separately.

4.2 Reduction to a system where every unknown appears twice

We consider a nonsingular irreducible system of Sylvester-like equations and we want to prove that the system can be reduced to another one in which each unknown appears exactly twice (and in different equations, when the system has at least two equations). For this purpose, we need the following result.

Theorem 11. *Let $\mathbb{S}$ be an irreducible system of equations in the form* (3) *with $r > 1$ equations and unknowns. If the unknown X_{α_k} appears in just one equation, say $A_k X_{\alpha_k}^{s_k} B_k - C_k X_{\beta_k}^{t_k} D_k = E_k$, then $\mathbb{S}$ is nonsingular if and only if A_k and B_k are invertible and the system $\widetilde{\mathbb{S}}$ formed by the remaining $r-1$ equations and unknowns is nonsingular.*

Proof. Note, first, that $\beta_k \neq \alpha_k$, and that the variable X_{β_k} appears again in $\widetilde{\mathbb{S}}$, otherwise $\mathbb{S}$ would be reducible. Suppose first that $\widetilde{\mathbb{S}}$ is nonsingular and A_k, B_k are invertible. Then, the unique solution of $\mathbb{S}$ is obtained by first solving $\widetilde{\mathbb{S}}$ to get the value of all the variables except X_{α_k}, and then computing X_{α_k} from

$$X_{\alpha_k}^{s_k} = A_k^{-1}(C_k X_{\beta_k}^{t_k} D_k + E_k) B_k^{-1}. \tag{11}$$

If $\widetilde{\mathbb{S}}$ has more than one solution, then this algorithm produces multiple solutions to $\mathbb{S}$. If $\widetilde{\mathbb{S}}$ has no solution, then clearly $\mathbb{S}$ has no solution either. If A_k is singular, let v be a nonzero vector such that $A_k v = 0$; then, given any solution to (3) we can replace $X_{\alpha_k}^{s_k}$ with $X_{\alpha_k}^{s_k} + vu^\top$, for any $u \in \mathbb{C}^n$, obtaining a new solution of (3), so $\mathbb{S}$ does not have a unique solution. A similar argument can be used if B_k is singular.

Moreover, $\widetilde{\mathbb{S}}$ is irreducible. Otherwise, it could be split in two systems with different unknowns, and just one of them would contain X_{β_k}; adding the kth equation to this last system would give a partition of the original system $\mathbb{S}$ in two systems with different unknowns. $\square$

The proof of Theorem 11 shows that, if an irreducible system $\mathbb{S}$ having $r > 1$ unknowns contains an unknown appearing just once in $\mathbb{S}$, then we can remove this unknown, together with its corresponding equation, to get a new irreducible system with $r-1$ equations and $r-1$ unknowns. Notice that the new system may have unknowns appearing just once, that can be removed if $r > 2$, using Theorem 11 again.

This elimination procedure can be repeated as long as the number of equations is greater than one and there is an unknown appearing just once. After a finite number of reductions (using Theorem 11 repeatedly), we arrive at an irreducible system $\widetilde{\mathbb{S}}$, which has the same number $\widetilde{r}$ of equations and unknowns and either $\widetilde{r} = 1$ or no unknown appears just once. In both cases, all unknowns in $\widetilde{\mathbb{S}}$ appear just twice. Moreover, $\mathbb{S}$ is nonsingular if and only if $\widetilde{\mathbb{S}}$ is nonsingular. Therefore, we can focus, from now on, on irreducible systems with the same number of equations and unknowns, and where each unknown appears exactly twice.

4.3 Reduction to a periodic system with at most one $\star$

In Section 4.2 we have proved that, without loss of generality, and regarding nonsingularity, we can consider irreducible systems of r Sylvester-like equations with r matrix unknowns, any of which appearing just twice. Now, we want to show that we can get an equivalent periodic system of the form (4) from any system of this form.

We first note that, by renaming the unknowns if necessary, under these assumptions the system (3) can be written in the form

$$\left\{ \begin{array}{rcl} A_k X_k^{s_k} B_k - C_k X_{k+1}^{t_k} D_k &=& E_k, \quad k = 1, \dots, r-1, \\ A_r X_r^{s_r} B_r - C_r X_1^{t_r} D_r &=& E_r, \end{array} \right. \tag{12}$$

where $s_k, t_k \in \{1, \star\}$. A way to show this is as follows. Let us start with X_1 and choose one of the two equations containing this unknown (there are at least two as long as the system contains at least two equations). Let this equation, with appropriate relabeling of the coefficients if needed, be $A_1 X_1^{s_1} B_1 - C_1 X_{\alpha_1}^{t_1} D_1 = E_1$. Now we look for the other equation containing X_{α_1}. With a relabeling of the coefficients if needed, this equation is $A_2 X_{\alpha_1}^{s_2} B_2 - C_2 X_{\alpha_2}^{t_2} D_2 = E_2$, and we proceed in this way with X_{α_2} and so on with the remaining unknowns. Note that, during this process, it cannot happen that $\alpha_i = \alpha_j$ for $i \neq j$, since otherwise X_{α_i} would appear more than twice in the system. Therefore, at some point we end up with $\alpha_t = 1$. If there were some $1 \leq j \leq r$ such that $j \neq \alpha_i$, for all $i = 1, \ldots, t$, then the system would be reducible. Hence, it must be $t = r$ and, by relabeling the unknowns as $\alpha_k = k+1$, for $k = 1, \ldots, r-1$, and $\alpha_r = 1$, we get the system in the form (12).

We now show that each periodic irreducible system of the form (12) can be reduced to the simpler form (4), with at most one $\star$. This can be obtained by applying a sequence of $\star$ operations and changes of variables, without further linear algebraic manipulations. This is stated in the following result.

Lemma 12. *Given a system of generalized $\star$-Sylvester equations* (12)*, there exists an equivalent system*

$$\begin{array}{rcl} \widetilde{A}_k Y_k \widetilde{B}_k - \widetilde{C}_k Y_{k+1} \widetilde{D}_k & = & \widetilde{E}_k, \quad k = 1, \ldots, r-1, \\ \widetilde{A}_r Y_r \widetilde{B}_r - \widetilde{C}_r Y_1^s \widetilde{D}_r & = & \widetilde{E}_r, \end{array} \tag{13}$$

which is obtained through a change of variables $Y_k = X_k^{u_k}$, with $u_k \in \{1, \star\}$.

Moreover, $s = 1$ if the number of $\star$ symbols appearing among s_i, t_i in the original system (12) *is even, and $s = \star$ if it is odd.*

Proof. The proof of this result is constructive, i.e., it directly provides an algorithm for the computation of the transformed system. Let m be the minimum index for which $(s_k, t_k) \neq (1, 1)$, with s_k, t_k being as in (12). If $(s_k, t_k) = (1, 1)$, for each $k = 1, \ldots, r$, then the minimum does not exist and we set $m = r+1$. If $m = r$ with $(s_m, t_m) = (1, \star)$ or $m = r+1$, then the system is already in the required form. The remaining cases are: $(s_m, t_m) = (\star, \star)$; $(s_m, t_m) = (\star, 1)$; and $(s_m, t_m) = (1, \star)$, with $m < r$.

If $s_m = t_m = \star$, then we can apply the $\star$ operator in the mth equation and go back to the case $s_m = t_m = 1$, since

$$A_m X_m^\star B_m - C_m X_{m+1}^\star D_m = E_m \iff B_m^\star X_m A_m^\star - D_m^\star X_{m+1} C_m^\star = E_m^\star.$$

The case $(s_m, t_m) = (\star, 1)$ can be turned into the case $(s_m, t_m) = (1, \star)$ by applying again the $\star$ operator in the mth equation. Finally, when $(s_m, t_m) = (1, \star)$ and $m < r$, we note that, by setting $Y_{m+1} = X_{m+1}^\star$, we have

$$A_m X_m B_m - C_m X_{m+1}^\star D_m = E_m \iff A_m X_m B_m - C_m Y_{m+1} D_m = E_m.$$

This change will require to update the exponent s_{m+1} in the $(m+1)$st equation (containing $X_{m+1} = Y_{m+1}^\star$). We can iterate the procedure until $m = r$ and $(s_m, t_m) = (1, \star)$ or $m = r+1$, then renaming the remaining unknowns X_k to Y_k we will transform the system into the required form (13). These transformations preserve the parity of the number of $\star$ symbols appearing within the equations, since each change of variables shifts the exponent, from $\star$ to 1 or viceversa, in the two appearances of each unknown. Therefore, the second part of the statement follows. ∎

The above results show that we can reduce the problem on the nonsingularity of (3) either to a periodic system of r generalized Sylvester equations or to a periodic system of $r-1$ generalized Sylvester and one generalized $\star$-Sylvester equation.

5 Reducing the problem to a block triangular linear system

In Section 4 we have seen how a system of general type (3) can be reduced to one or more periodic systems of the type (4), where all equations are generalized Sylvester equations except the last one, that is either a generalized Sylvester or a generalized $\star$-Sylvester equation.

Here we focus on a periodic system of type (4). First, we show in Section 5.1 that it can be transformed into an equivalent periodic system with triangular coefficients. Then, in Section 5.2 we show that, in the cases $s = 1$ and $s = \top$, the latter system is a linear system whose coefficient matrix is block triangular with diagonal blocks of order r or $2r$. Finally, in Section 5.3 we show that the case $s = *$ can be reduced to the case $s = 1$.

The reduction to a special linear system allows us to deduce useful conditions for the nonsingularity of a system of generalized Sylvester equations and, moreover, to design an efficient numerical algorithm for its solution.

5.1 Reduction to a system with triangular coefficients

We can multiply by suitable unitary matrices and perform a change of variables on the system (4) so that the matrices A_k, B_k, C_k, D_k are all upper or lower (quasi-)triangular.

Lemma 13. *There exists a change of variables of the form $\widehat{X}_k = Z_k^* X_k \widehat{Z}_k$, with $Z_k, \widehat{Z}_k \in \mathbb{C}^{n\times n}$ unitary, for $k = 1, 2, \dots, r$, which makes simultaneously the coefficients A_k, C_k of (4) upper triangular, and the coefficients B_k, D_k lower triangular, after pre-multiplying and post-multiplying the kth equation by appropriate unitary matrices Q_k and $\widehat{Q}_k$, respectively.*

Proof. Let us start with the case $s = 1$. This case is treated in [5], but we report the algorithm for completeness. Let

$$Q_k^* A_k Z_k = \widehat{A}_k, \quad Q_k^* C_k Z_{k+1} = \widehat{C}_k,$$

with $\widehat{A}_k, \widehat{C}_k$ upper triangular, be a periodic Schur form of the formal matrix product $C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1$, and

$$\widehat{Q}_k^* B_k^* \widehat{Z}_k = \widehat{B}_k^*, \quad \widehat{Q}_k^* D_k^* \widehat{Z}_{k+1} = \widehat{D}_k^*,$$

with $\widehat{B}_k^*, \widehat{D}_k^*$ upper triangular, be a periodic Schur form of the formal matrix product $D_r^{-*} B_r^* D_{r-1}^{-*} B_{r-1}^* \cdots D_1^{-*} B_1^*$ (see Section 2). Setting $\widehat{X}_k = Z_k^* X_k \widehat{Z}_k$ and $\widehat{E}_k = Q_k^* E_k \widehat{Q}_k$ for all k, the equations of (4) are equivalent to

$$\begin{aligned}
\widehat{A}_k \widehat{X}_k \widehat{B}_k - \widehat{C}_k \widehat{X}_{k+1} \widehat{D}_k &= Q_k^* A_k Z_k \widehat{X}_k \widehat{Z}_k^* B_k \widehat{Q}_k - Q_k^* C_k Z_{k+1} \widehat{X}_{k+1} \widehat{Z}_{k+1}^* D_k \widehat{Q}_k \\
&= Q_k^* (A_k X_k B_k - C_k X_{k+1} D_k) \widehat{Q}_k \\
&= Q_k^* E_k \widehat{Q}_k = \widehat{E}_k,
\end{aligned}$$

where $X_{r+1} = X_1$ and $\widehat{X}_{r+1} = \widehat{X}_1$. Hence we get equations in the same form as (4), but with triangular coefficients.

We now deal with the case $s = *$. Let

$$\begin{aligned} Q_k^* A_k Z_k &= \widehat{A}_k, & Q_k^* C_k Z_{k+1} &= \widehat{C}_k, & Z_{2r+1} &= Z_1, \\ Q_{r+k}^* B_k^* Z_{r+k} &= \widehat{B}_k^*, & Q_{r+k}^* D_k^* Z_{r+k+1} &= \widehat{D}_k^*, & k &= 1, 2, \dots, r, \end{aligned}$$

be a periodic Schur form of the formal matrix product

$$D_r^{-*} B_r^* D_{r-1}^{-*} B_{r-1}^* \cdots D_1^{-*} B_1^* C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1.$$

Setting $\widehat{X}_k = Z_k^* X_k Z_{r+k}$ and $\widehat{E}_k = Q_k^* E_k Q_{r+k}$, for all $k = 1, \dots, r$, the first $r-1$ equations of (4) are equivalent to

$$\begin{aligned} \widehat{A}_k \widehat{X}_k \widehat{B}_k - \widehat{C}_k \widehat{X}_{k+1} \widehat{D}_k &= Q_k^* A_k Z_k \widehat{X}_k Z_{r+k}^* B_k Q_{r+k} - Q_k^* C_k Z_{k+1} \widehat{X}_{k+1} Z_{r+k+1}^* D_k Q_{r+k} \\ &= Q_k^* (A_k X_k B_k - C_k X_{k+1} D_k) Q_{r+k} \\ &= Q_k^* E_k Q_{r+k} = \widehat{E}_k, \end{aligned}$$

and the last one is equivalent to

$$\begin{aligned} \widehat{A}_r \widehat{X}_r \widehat{B}_r - \widehat{C}_r \widehat{X}_1^* \widehat{D}_r &= Q_r^* A_r Z_r \widehat{X}_r Z_{2r}^* B_r Q_{2r} - Q_r^* C_r Z_{r+1} \widehat{X}_1^* Z_1^* D_r Q_{2r} \\ &= Q_r^* (A_r X_r B_r - C_r X_1^* D_r) Q_{2r} \\ &= Q_r^* E_r Q_{2r} = \widehat{E}_r. \end{aligned}$$

Hence we get once again the same shape as (4) and triangular coefficients.

Finally, let us treat the case $s = \top$. Let

$$\begin{aligned} Q_k^* A_k Z_k &= \widehat{A}_k, & Q_k^* C_k Z_{k+1} &= \widehat{C}_k, & Z_{2r+1} &= Z_1, \\ Q_{r+k}^* B_k^\top Z_{r+k} &= \widehat{B}_k^\top, & Q_{r+k}^* D_k^\top Z_{r+k+1} &= \widehat{D}_k^\top, & k &= 1, 2, \dots, r, \end{aligned}$$

be a periodic Schur form of the formal matrix product

$$D_r^{-\top} B_r^\top D_{r-1}^{-\top} B_{r-1}^\top \cdots D_1^{-\top} B_1^\top C_r^{-1} A_r C_{r-1}^{-1} A_{r-1} \cdots C_1^{-1} A_1.$$

We set $\widehat{X}_k = Z_k^* X_k \overline{Z}_{r+k}$, where $\overline{Z}_{r+k} = ((Z_{r+k})^*)^\top$ denotes the elementwise conjugate (without transposition) of Z_{r+k}, and $\widehat{E}_k = Q_k^* E_k \overline{Q}_{r+k}$. The first $r-1$ equations of (4) are equivalent to

$$\begin{aligned} \widehat{A}_k \widehat{X}_k \widehat{B}_k - \widehat{C}_k \widehat{X}_{k+1} \widehat{D}_k &= Q_k^* A_k Z_k \widehat{X}_k Z_{r+k}^\top B_k \overline{Q}_{r+k} - Q_k^* C_k Z_{k+1} \widehat{X}_{k+1} Z_{r+k+1}^\top D_k \overline{Q}_{r+k} \\ &= Q_k^* (A_k X_k B_k - C_k X_{k+1} D_k) \overline{Q}_{r+k} \\ &= Q_k^* E_k \overline{Q}_{r+k} = \widehat{E}_k, \end{aligned}$$

and the last one is equivalent to

$$\begin{aligned} \widehat{A}_r \widehat{X}_r \widehat{B}_r - \widehat{C}_r \widehat{X}_1 \widehat{D}_r &= Q_r^* A_r Z_r \widehat{X}_r Z_{2r}^\top B_r \overline{Q}_{2r} - Q_r^* C_r Z_{r+1} \widehat{X}_1^\top Z_1^\top D_r \overline{Q}_{2r} \\ &= Q_r^* (A_r X_r B_r - C_r X_1^\top D_r) \overline{Q}_{2r} \\ &= Q_r^* E_r \overline{Q}_{2r} = \widehat{E}_r. \end{aligned}$$

□

5.2 Reduction to a block upper triangular linear system for $s = 1$ and $s = \top$

A system like (4) can be seen as a system of $n^2 r$ equations in $n^2 r$ unknowns in terms of the entries of the unknown matrices. This is a linear system for $s = 1$ or $s = \top$, while in the case $\star = *$ it is not linear over $\mathbb{C}$ due to the conjugation. Nevertheless, it can be either transformed into a linear system over $\mathbb{R}$, by splitting the real and imaginary parts of both the coefficient matrices and the unknowns, or into a linear system over $\mathbb{C}$ by doubling the size (see Section 5.3).

A standard approach to get explicitly the matrix coefficient of the (linear) system associated with a system of Sylvester-like equations is to exploit the relation $\operatorname{vec}(AXB) = (B^\top \otimes A) \operatorname{vec} X$ [17, Lemma 4.3.1] where the $\operatorname{vec}(\cdot)$ operator maps a matrix into the vector obtained by stacking its columns one on top of the other, and $A \otimes B$ is the Kronecker product of A and B, namely the block matrix with blocks of the type $[a_{ij}B]$ (see [17, Ch. 4]).

Relying on the reduction scheme that we have presented in Section 5.1, we may assume that the coefficients A_k, C_k, and B_k, D_k, in (4) are upper and lower triangular matrices, respectively. In this case the matrix of the linear system obtained after applying the $\operatorname{vec}(\cdot)$ operator has a nice structure; indeed, performing appropriate row and column permutations to the matrix (in other words, choosing an appropriate ordering of the unknowns), in Section 5.2.1, we get a block upper triangular coefficient matrix, with diagonal blocks of order r or $2r$, and thus much more manageable.

In the case where $s = 1$, a characterization for nonsingularity was obtained in [5] (see Theorem 7). The approach followed in that reference is similar to the one we follow here.

We first deal with the cases $s \in \{1, \top\}$, which are both linear, and for which we can directly give conditions based on the matrix representing the linear system in the entries of the unknowns. This is the aim of Section 5.2.1. The case $s = *$ can be reduced to the case $s = 1$ using specific developments which are contained in Section 5.3.

5.2.1 Making the matrix coefficient block triangular

We assume that A_k, C_k are upper triangular and B_k, D_k are lower triangular, for $k = 1, \ldots, r$.

Using the relation $\operatorname{vec}(AXB) = (B^\top \otimes A) \operatorname{vec} X$ we can rewrite the system (4), for the case $s = 1$, as the linear system

$$\begin{bmatrix} B_1^\top \otimes A_1 & -D_1^\top \otimes C_1 & & \\ & \ddots & \ddots & \\ & & B_{r-1}^\top \otimes A_{k-1} & -D_{r-1}^\top \otimes C_{r-1} \\ -D_r^\top \otimes C_r & & & B_r^\top \otimes A_r \end{bmatrix} \mathcal{X} = \mathcal{E}, \qquad (14)$$

where the empty block entries should be understood as zero blocks, and

$$\mathcal{X} := \begin{bmatrix} \operatorname{vec} X_1 \\ \vdots \\ \operatorname{vec} X_r \end{bmatrix}, \qquad \mathcal{E} := \begin{bmatrix} \operatorname{vec} E_1 \\ \vdots \\ \operatorname{vec} E_r \end{bmatrix}.$$

In the case $s = \top$ we have, instead

$$\begin{bmatrix} B_1^\top \otimes A_1 & -D_1^\top \otimes C_1 & & & \\ & \ddots & \ddots & & \\ & & & B_{r-1}^\top \otimes A_{k-1} & -D_{r-1}^\top \otimes C_{r-1} \\ -(D_r^\top \otimes C_r)P_{n,n} & & & & B_r^\top \otimes A_r \end{bmatrix} \mathcal{X} = \mathcal{E}, \tag{15}$$

where $P_{a,b}$ denotes the *commutation matrix*, i.e., the permutation matrix such that $P_{a,b} \operatorname{vec} X = \operatorname{vec}(X^\top)$ for each $X \in \mathbb{R}^{a\times b}$ [17, Th. 4.3.8].

In the following, we will index the components of $\mathcal{X}$ by means of the triple (i, j, k), that denotes the (i, j) entry of X_k. This is just a shorthand for the component $(k-1)n^2 + (j-1)n + i$ of $\mathcal{X}$.

Notice that each coordinate of any of the systems (14) and (15) can be obtained by left multiplying one of the r equations of (4) by $e_i^\top$ on the left and by e_j on the right, for appropriate $1 \le i, j \le n$.

We are interested in performing a permutation on systems (14) and (15) that takes them to block upper triangular form (independently on the presence of the permutation matrix P). In order to achieve this goal the choice is not unique, and different choices have different advantages. For this reason, we first characterize a set of permutations from which we will choose one.

Definition 14. We say that an ordering $\preceq_S$ on $\{1, \dots, n\}^2$ is *echelon-shaped* if it satisfies the following two conditions:

(a) For any pairs (i, j) and (i', j') such that $\min\{i, j\} \le \min\{i', j'\}$ and $\max\{i, j\} \le \max\{i', j'\}$, either $(i, j) \preceq_S (i', j')$ or $(i', j') = (j, i)$;

(b) (i, j) and (j, i) are always consecutive, i.e.

$$(i, j) \preceq_S (i', j') \preceq_S (j, i) \implies (i', j') \in \{(i, j), (j, i)\}.$$

Figure 1 illustrates the meaning of Definition 14 over the set of indices of a square matrix. An order is echelon-shaped if all the indices of the $\circ$ entries are smaller than or equal the ones of the $\times$ entries, and the ordering between the indices of the two $\times$ does not matter, as long as they are consecutive.

Our interest in echelon-shaped orderings is motivated by the following result.

Lemma 15. *Let A_k, C_k be $n \times n$ upper triangular matrices and B_k, D_k be $n \times n$ lower triangular matrices, for $k = 1, \dots, r$. Let $\preceq$ be an order on (i, j, k) such that there exists an echelon-shaped ordering $\preceq_S$ satisfying*

$$\text{“}(i, j, k) \preceq (i', j', k') \qquad \implies \qquad (i, j) \preceq_S (i', j'),\text{”}.$$

$$\begin{bmatrix} & & & \times & \circ & \circ & \circ \\ & & & \circ & \circ & \circ & \circ \\ & & & \circ & \circ & \circ & \circ \\ \times & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{bmatrix}$$

Figure 1: Representation of a echelon-shaped ordering. The $\circ$ elements are all larger than the $\times$ ones. The two $\times$ ones must be consecutive.

Let $\mathbb{S}$ be the system of $n^2 r$ equations

$$\begin{cases} e_i^\top (A_k X_k B_k - C_k X_{k+1} D_k) e_j &= (E_k)_{ij}, \quad i,j = 1,\dots,n,\ k = 1,\dots,r-1, \\ e_i^\top (A_r X_r B_r - C_r X_1^s D_r) e_j &= (E_r)_{ij}, \quad\quad\quad i,j = 1,\dots,n, \end{cases} \tag{16}$$

in the $n^2 r$ unknowns x_{ijk}, $i,j = 1,\dots,n$ and $k = 1,\dots,r$, where x_{ijk} is the (i,j) entry of X_k. If both the set of indices (i,j,k) of the equations in $\mathbb{S}$ and the unknowns are ordered in increasing order with respect to $\preceq$, then the coefficient matrix associated with $\mathbb{S}$ is block upper triangular, with blocks of size either $r \times r$ or $2r \times 2r$.

Proof. We block partition the matrix associated with the system $\mathbb{S}$, so that the equations and the unknowns in $\mathbb{S}$ with indices in

$$\mathcal{S}_{ij} := \{(i,j,1),\dots,(i,j,r)\} \cup \{(j,i,1),\dots,(j,i,r)\}$$

correspond to a block with size $r \times r$ if $i = j$ or $(2r) \times (2r)$ if $i \neq j$. Notice that the position of each block in the whole matrix depends on the ordering $\preceq_S$ and thus the matrix is block upper triangular if the equations with indices in $\mathcal{S}_{ij}$ contain only unknowns with indices (i', j', k) of the form $(i,j) = (j', i')$ or $(i,j) \preceq_S (i', j')$.

Let us first consider an equation in $\mathbb{S}$ with index (i,j,k), and $1 \leq k \leq r-1$. From (16), and taking into account that A_k, C_k are upper triangular and B_k, D_k are lower triangular, this equation involves unknowns with indices (i', j', k) and $(i', j', k+1)$ satisfying $i \leq i'$ and $j \leq j'$. Since $\preceq_S$ is echelon-shaped, this implies $(i,j) \preceq_S (i', j')$.

Now, let us consider an equation in $\mathbb{S}$ with index (i,j,r). From (16), and using again that A_r, C_r are upper triangular and B_r, D_r are lower triangular, this equation involves unknowns (i', j', r) with $i \leq i'$ and $j \leq j'$, and also unknowns $(i', j', 1)$ with either $(i,j) = (j', i')$ or $i \leq j'$ and $j \leq i'$. Again, since $\preceq_S$ is echelon-shaped, this implies that $(i,j) \preceq_S (i', j')$. □

Remark 16. Looking at the proof of Lemma 15, one realizes that, if $X_1^\top$ in the last equation is replaced by X_1, then it would be enough that the ordering $\preceq_S$

is such that $(i,j) \preceq_S (s,t)$ whenever $i \leq s$ and $j \leq t$ to get the conclusion in the statement. Then, the echelon-shaped condition of $\preceq_S$ is essentially related to the last equation, which involves $X_1^\top$.

Remark 17. The ordering on the indices (i,j,k) obtained naturally by vectorizing the matrix unknowns one after the other, i.e.,

$$\mathcal{X} = \begin{bmatrix} \operatorname{vec} X_1 \\ \operatorname{vec} X_2 \\ \vdots \\ \operatorname{vec} X_r \end{bmatrix},$$

is the order $\preceq_1$ given by: $(i,j,k) \preceq_1 (i',j',k')$ if and only if:

(a) $k < k'$, or

(b) $k = k'$ and

  (b1) $j < j'$, or

  (b2) $j = j'$ and $i \leq i'$,

which is usually called *reverse lexicographic order*. Even though this ordering is natural when considering vectorized matrices and tensors, it does not fulfill the hypotheses in Lemma 15.

We now suggest two examples of echelon-shaped orderings. They are illustrated in Figure 2 for the entries of a 4×4 matrix.

Definition 18. We say that $(i,j) \preceq_{RC} (i',j')$ if and only if one of the following conditions hold:

(a) $\max\{i,j\} < \max\{i',j'\}$, or

(b) $\max\{i,j\} = \max\{i',j'\}$ and $\min\{i,j\} < \min\{i',j'\}$, or

(c) $(i',j') = (j,i)$ and $i \leq j$.

We refer to $\preceq_{RC}$ as the *row-column-ordering*, since it traverses the matrices one column and one row at a time.

Definition 19. We say that $(i,j) \preceq_A (i',j')$ if and only if one of the following conditions hold:

(a) $i + j < i' + j'$, or

(b) $i + j = i' + j'$ and $\max\{i,j\} < \max\{i',j'\}$, or

(c) $(i,j) = (j',i')$ and $i \leq j$.

We refer to $\preceq_A$ as the *antidiagonal-ordering*, since it traverses the matrix one antidiagonal at a time.

$$\begin{bmatrix} 1 & 2 & 5 & 10 \\ 3 & 4 & 7 & 12 \\ 6 & 8 & 9 & 14 \\ 11 & 13 & 15 & 16 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 5 & 9 \\ 3 & 4 & 7 & 12 \\ 6 & 8 & 11 & 14 \\ 10 & 13 & 15 & 16 \end{bmatrix}$$

Figure 2: Representation of the row-column-ordering $\preceq_{RC}$ for (i, j, k) (on the left), and of the antidiagonal-ordering $\preceq_A$ (on the right). The numbers indicate the order of the corresponding entries.

Lemma 20. *Both $\preceq_A$ and $\preceq_{RC}$ are echelon-shaped orderings. In particular, they satisfy the hypotheses of Lemma* 15*, so that the matrix of the linear system $\mathbb{S}$, with the variables ordered according to $\preceq_A$ or $\preceq_{RC}$, is block upper triangular.*

Proof. The statement can be verified by applying Definition 14 directly. □

The above orderings will be used in Section 6 to prove Theorems 4 and 5. Indeed, a necessary and sufficient condition for the nonsingularity of a periodic system of Sylvester-like equations (4), with $s = 1, \top$, is the invertibility of the ($r \times r$ or $(2r) \times (2r)$) diagonal blocks of the matrix associated with the large $(n^2 r) \times (n^2 r)$ linear system (16).

The same orderings are also the main tool used in Section 7 to develop an algorithm for the solution of the large $(n^2 r) \times (n^2 r)$ linear system in a fast way (more precisely, in $O(n^3 r)$ flops).

5.2.2 Characterizing the diagonal blocks

Both from the computational and from the theoretical point of view we are interested in characterizing the structure of the diagonal blocks of the coefficient matrix associated with the permuted linear system.

Theoretically, this is interesting because the system (4) is nonsingular if and only if the determinants of all diagonal blocks are nonzero. This will allow us to prove Theorems 4 and 5.

Computationally, this is relevant because these are the matrices that allow one to carry out the block back substitution process to compute the solution of (4), when it is unique.

As already pointed out in Section 5.2.1 the diagonal blocks can be obtained by choosing a pair (i, j) and selecting the equations given by

$$\begin{aligned} e_i^\top (A_k X_k B_k - C_k X_{k+1} D_k) e_j &= (E_k)_{ij}, \quad k = 1, \dots, r-1, \\ e_i^\top (A_r X_r B_r - C_r X_1^s D_r) e_j &= (E_r)_{ij}, \end{aligned}$$

and removing all the variables with indices different from (i, j) and (j, i). As mentioned in the proof of Lemma 15, these other variables have indices (i', j', k') with $(i, j) \preceq_S (i', j')$, and $\preceq_S$ being an echelon-shaped ordering like the ones described at the end of Section 5.2.1. When $i = j$ this gives us an $r \times r$ linear system, otherwise we obtain a $2r \times 2r$ linear system.

Notice that this procedure can be carried out both in the case $s \in \{1, \top\}$ and in the $s = *$ case, even if in the latter these systems are nonlinear. We denote them with $\mathbb{S}_{ij} \cup \mathbb{S}_{ji}$ or simply with $\mathbb{S}_{ii}$ when $i = j$, according to the notation used for the indices in the proof of Lemma 15.

Luckily, if we choose an ordering, $\prec$, such that $(i, j, k) \prec (i, j, k')$ for $k < k'$, then all the systems share the same simple structure, so we can rely on the following result to compute the determinant of the coefficient matrix.

Lemma 21. *The matrix M defined as follows*

$$M = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{s-1} \\ \beta_s & & & \alpha_s \end{bmatrix}$$

has determinant equal to $\det M = \prod_{k=1}^{s} \alpha_k - (-1)^s \prod_{k=1}^{s} \beta_k$.

Proof. Use Laplace's determinant expansion on the first column. □

In the cases $s \in \{1, \top\}$, $\mathbb{S}_{ii}$ is an $r \times r$ linear system in the variables $(X_1)_{ii}, \ldots, (X_r)_{ii}$ with coefficient matrix:

$$M_{ii} := \begin{bmatrix} (A_1)_{ii}(B_1)_{ii} & -(C_1)_{ii}(D_1)_{ii} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{ii} & -(C_{r-1})_{ii}(D_{r-1})_{ii} \\ -(C_r)_{ii}(D_r)_{ii} & & & (A_r)_{ii}(B_r)_{ii} \end{bmatrix}. \tag{17}$$

According to Lemma 21 we have:

$$\det M_{ii} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{ii} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{ii}\,. \tag{18}$$

A similar relation holds also when $i \neq j$ in the $s = 1$ case, since $\mathbb{S}_{ij}$ and $\mathbb{S}_{ji}$ are decoupled systems. More precisely, the coefficient matrix of $\mathbb{S}_{ij}$ in the case $s = 1$ is

$$M_{ij} := \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & (A_{r-1})_{ii}(B_{r-1})_{jj} & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ -(C_r)_{ii}(D_r)_{jj} & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}, \tag{19}$$

and similarly for the one of $\mathbb{S}_{ji}$ exchanging the roles of i and j. From Lemma 21 we get:

$$\det M_{ij} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{jj} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{jj}. \tag{20}$$

In the case $s = \top$, instead, the systems $\mathbb{S}_{ij} \cup \mathbb{S}_{ji}$ form a $2r \times 2r$ linear system in the variables $(X_k)_{ij}$, $(X_k)_{ji}$, for $k = 1, \dots, r$, with coefficient matrix

$$M_{ij} := \begin{bmatrix} \mathcal{B}_{ij} & -(C_r)_{ii}(D_r)_{jj} e_r e_1^\top \\ -(C_1)_{jj}(D_1)_{ii} e_r e_1^\top & \mathcal{B}_{ji} \end{bmatrix}, \tag{21}$$

where

$$\mathcal{B}_{ij} = \begin{bmatrix} (A_1)_{ii}(B_1)_{jj} & -(C_1)_{ii}(D_1)_{jj} & & \\ & \ddots & \ddots & \\ & & \ddots & -(C_{r-1})_{ii}(D_{r-1})_{jj} \\ & & & (A_r)_{ii}(B_r)_{jj} \end{bmatrix}.$$

Thanks, again, to Lemma 21, this matrix has determinant equal to

$$\det M_{ij} = \prod_{k=1}^{r} (A_k)_{ii}(B_k)_{ii}(A_k)_{jj}(B_k)_{jj} - \prod_{k=1}^{r} (C_k)_{ii}(D_k)_{ii}(C_k)_{jj}(D_k)_{jj}. \tag{22}$$

5.3 Linearizing the case $s = *$

We have already mentioned that, when $s = *$, the system (4) is not linear over the complex field, since it involves not only the entries of the matrix X_1 but also their conjugates. A method to transform it into a linear system over $\mathbb{C}$ is as follows: in addition to the equations of the system, we consider the equations obtained by taking their conjugate transpose, namely

$$\begin{array}{rcll} B_k^* X_k^* A_k^* - D_k^* X_{k+1}^* C_k^* & = & E_k^*, & k = 1, \dots, r-1, \\ B_r X_r^* A_r - D_r^* X_1 C_r^* & = & E_r^*. & \end{array}$$

If we consider X_k and X_k^* as two separate variables, then this is a system of $2r$ generalized Sylvester equations in $2r$ matrix unknowns. We prove more formally that this process produces an equivalent system.

Lemma 22. *The system* (4) *is nonsingular if and only if the system*

$$\left\{ \begin{array}{rcll} A_k X_k B_k - C_k X_{k+1} D_k & = & E_k, & k = 1, \dots, r-1, \\ A_r X_r B_r - C_r X_{r+1} D_r & = & E_r, & \\ B_k^* X_{r+k} A_k^* - D_k^* X_{r+k+1} C_k^* & = & E_k^*, & k = 1, \dots, r-1, \\ B_r^* X_{2r} A_r^* - D_r^* X_1 C_r^* & = & E_r^* & \end{array} \right. \tag{23}$$

is nonsingular.

Proof. We may consider only the case in which $E_k = 0$: checking nonsingularity corresponds to checking that there are no solutions to this homogenous system apart from the trivial one $X_k = 0$, for $k = 1, \dots, r$.

Let us first assume that (4) has a nonzero solution $(X_1, \dots, X_r)$. Then $(X_1, \dots, X_r, X_1^*, \dots, X_r^*)$ is a nonzero solution of (23).

Conversely, if $(X_1, \dots, X_r, X_{r+1}, \dots, X_{2r})$ is a nonzero solution of (23), then $(X_1 + X_{r+1}^*, \dots, X_r + X_{2r}^*)$ is a solution of (4). If $(X_1 + X_{r+1}^*, \dots, X_r + X_{2r}^*) = 0$, then $X_{r+i} = -X_i^*$, for $i = 1, \dots, r$, and then $\mathfrak{i}(X_1, \dots, X_r)$ is a nonzero solution of (4). ☐

Remark 23. The proof of Lemma 22 does not work if one replaces $*$ with $\top$ everywhere: it breaks in the final part, because $\mathfrak{i}(X_1, \dots, X_r)$ is not necessarily a solution of (4) with $\star = \top$. Indeed, Lemma 22 is false with $\top$ instead of $*$. Let us consider, for instance, the case $n = r = 1$ and the equation $x_1 + x_1^\top = 2x_1 = 0$. This equation has only the trivial solution, but the linearized system

$$\begin{cases} z_1 + z_2 = 0 \\ z_1 + z_2 = 0 \end{cases}$$

has infinitely many solutions.

Another relevant difference between the $\star = \top$ and the $\star = *$ cases is the following. System (4) is nonsingular if and only if the system obtained after replacing the minus sign in the last equation by a plus sign

$$\left\{ \begin{array}{rcl} A_k X_k B_k - C_k X_{k+1} D_k & = & E_k, \quad k = 1, \dots, r-1, \\ A_r X_r B_r + C_r X_1^* D_r & = & E_r \end{array} \right. \tag{24}$$

is nonsingular. To see this, reduce again to the case $E_k = 0$ for all $k = 1, \dots, r$ and note that if $(X_1, \dots, X_r)$ is a nonzero solution of (4) then $\mathfrak{i}(X_1, \dots, X_r)$ is a nonzero solution of (24), and viceversa. This property no longer holds true with $s = \top$.

6 Proofs of the main results

We first prove Theorems 4–5.

Proof of Theorem 4. We can consider only the case in which $E_i = 0$, $i = 1, 2, \dots, r$. Using the periodic Schur form of the formal products (8) we may consider the equivalent system (see the proof of Lemma 13)

$$\left\{ \begin{array}{rcl} \widehat{A}_k X_k \widehat{B}_k - \widehat{C}_k X_{k+1} \widehat{D}_k & = & 0, \quad k = 1, \dots, r-1, \\ \widehat{A}_r X_r \widehat{B}_r - \widehat{C}_r X_1 \widehat{D}_r & = & 0, \end{array} \right.$$

where, for each k, the matrices $\widehat{A}_k$ and $\widehat{C}_k$ are upper triangular and $\widehat{B}_k$ and $\widehat{D}_k$ are lower triangular. If the formal products (8) are regular, then their eigenvalues are the ratios $\lambda_i := \prod_{k=1}^r \frac{(\widehat{A}_k)_{ii}}{(\widehat{C}_k)_{ii}}$, $\mu_i := \prod_{k=1}^r \frac{(\widehat{D}_k)_{ii}}{(\widehat{B}_k)_{ii}}$, respectively, for $i = 1, \dots, n$ (they are allowed to be ∞).

With this triangularity assumption, in Lemma 15 we have shown that the system of Sylvester equations is equivalent to a block upper triangular system whose matrix coefficient has determinant $\delta := \prod_{i,j=1}^n \det(M_{ij})$, where M_{ij} is defined in (17) and (19).

In summary, the system of Sylvester equations is nonsingular if and only if $\delta \neq 0$, which, using (18) and (20), is equivalent to requiring

$$\prod_{k=1}^{r} (\widehat{A}_k)_{ii} (\widehat{B}_k)_{jj} \neq \prod_{k=1}^{r} (\widehat{C}_k)_{ii} (\widehat{D}_k)_{jj}, \qquad i, j = 1, \dots, n. \tag{25}$$

If $\delta \neq 0$, then it cannot happen that $\prod_k (\widehat{A}_k)_{ii}$ and $\prod_k (\widehat{C}_k)_{ii}$ are both zero or that $\prod_k (\widehat{B}_k)_{ii}$ and $\prod_k (\widehat{D}_k)_{ii}$ are both zero and thus the formal products are regular. Moreover, condition (25) implies that $\lambda_i \neq \mu_j$ for any $i, j = 1, \dots, n$ and thus the two products have disjoint spectra.

On the contrary, if $\delta = 0$ then the equality holds in (25) for some i and j. One can check that this condition implies that either one of the two formal products is singular or $\lambda_i = \mu_j$ and they cannot have disjoint spectra. □

We now give the proof of Theorem 5 separating the cases $\star = \top$ and $\star = *$ since the techniques we use are different.

Proof of Theorem 5 *for* $\star = \top$. Proceeding as in the proof of Theorem 4, we use the periodic Schur form of the formal product (9) and we get the equivalent system (see the proof of Lemma 13)

$$\begin{cases} \widehat{A}_k X_k \widehat{B}_k - \widehat{C}_k X_{k+1} \widehat{D}_k &= 0, \quad k = 1, \dots, r-1, \\ \widehat{A}_r X_r \widehat{B}_r - \widehat{C}_r X_1^\top \widehat{D}_r &= 0, \end{cases}$$

where, for each k, the matrices $\widehat{A}_k$ and $\widehat{C}_k$ are upper triangular and $\widehat{B}_k$ and $\widehat{D}_k$ are lower triangular. If the formal product (8) is regular, then its eigenvalues are the ratios $\lambda_i := \prod_{k=1}^{r} \frac{(\widehat{A}_k)_{ii} (\widehat{B}_k)_{ii}}{(\widehat{C}_k)_{ii} (\widehat{D}_k)_{ii}}$, for $i = 1, \dots, n$.

With this triangularity assumption, in Lemma 15 we have shown that the system of Sylvester equations is equivalent to a block upper triangular system whose matrix coefficient has determinant $\delta := \prod_{i=1}^{n} \det(M_{ii}) \prod_{\substack{i,j=1 \\ i<j}}^{n} \det(M_{ij})$, where M_{ii} is defined in (17) and M_{ij}, for $i \neq j$, in (21).

In summary, the system of Sylvester-like equations is nonsingular if and only if $\delta \neq 0$, that, using (18) and (22), is equivalent to requiring

$$\begin{aligned} &\prod_{k=1}^{r} (\widehat{A}_k)_{ii} (\widehat{B}_k)_{ii} \neq \prod_{k=1}^{r} (\widehat{C}_k)_{ii} (\widehat{D}_k)_{ii}\,, && i = 1, \dots, n, \\ &\prod_{k=1}^{r} (\widehat{A}_k)_{ii} (\widehat{B}_k)_{ii} (\widehat{A}_k)_{jj} (\widehat{B}_k)_{jj} \neq \prod_{k=1}^{r} (\widehat{C}_k)_{ii} (\widehat{D}_k)_{ii} (\widehat{C}_k)_{jj} (\widehat{D}_k)_{jj}\,, && i \neq j. \end{aligned} \tag{26}$$

If $\delta \neq 0$, then it cannot happen that $\prod_k (\widehat{A}_k)_{ii} (\widehat{B}_k)_{ii}$ and $\prod_k (\widehat{C}_k)_{ii} (\widehat{D}_k)_{ii}$ are both zero, for some i, thus the formal product (8) is regular. Moreover, conditions (26) imply that

$$\begin{cases} \lambda_i \neq 1, & i = 1, \dots, n \\ \lambda_i \neq \lambda_j^{-1}, & i \neq j, \end{cases}$$

and this implies in turn that the spectrum $\Lambda(\Pi)\setminus\{-1\}$ is reciprocal free and the multiplicity of $\{-1\}$ is at most one.

On the contrary, if $\delta = 0$ then the equality holds in (26) above for some i or below for some couple (i,j), with $i \neq j$. One can check that this condition implies that one of the following cases holds: (a) the formal product is singular; (b) $\lambda_i = 1$, for some i, and thus $\Lambda(\Pi)\setminus\{-1\}$ is not reciprocal free; (c) $\lambda_i = 1/\mu_j \neq -1$, for some $i \neq j$, and thus $\Lambda(\Pi)\setminus\{-1\}$ is not reciprocal free; (d) $\lambda_i = 1/\mu_j = -1$ and the multiplicity of -1 is greater than 1. □

Using Lemma 22, the following argument allows us to obtain Theorem 5 with $\star = *$ directly as a consequence of Theorem 4.

Proof of Theorem 5 *for* $\star = *$. Let us start from a system of the form (4) with $s = *$. Lemma 22 shows that it is nonsingular if and only if the larger linear system (23) is nonsingular. System (23) is a system of $2r$ generalized Sylvester equations with $s = 1$. Hence we can apply Theorem 4 to this system, obtaining that (23) is nonsingular if and only if the two formal products

$$\Pi_1 := \Pi = D_r^{-*}B_r^*D_{r-1}^{-*}B_{r-1}^*\cdots D_1^{-*}B_1^*C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1$$

and

$$\Pi_2 := C_r^*A_r^{-*}C_{r-1}^*A_{r-1}^{-*}\cdots C_1^*A_1^{-*}D_rB_r^{-1}D_{r-1}B_{r-1}^{-1}\cdots D_1B_1^{-1}$$

are regular and have no common eigenvalues. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of Π_1, then the eigenvalues of the formal product

$$\Pi_2^{-*} := C_r^{-1}A_rC_{r-1}^{-1}A_{r-1}\cdots C_1^{-1}A_1D_r^{-*}B_r^*D_{r-1}^{-*}B_{r-1}^*\cdots D_1^{-*}B_1^*$$

are again $\lambda_1, \lambda_2, \ldots, \lambda_n$, because Π_2^{-*} differs from Π_1 only by a cyclic permutation of the factors. This proves that the eigenvalues of Π_2 are $(\overline{\lambda}_1)^{-1}, (\overline{\lambda}_2)^{-1}, \ldots, (\overline{\lambda}_n)^{-1}$, so they are distinct from those of Π_1 if and only if $\Lambda(\Pi_1)$ is a $*$-reciprocal free set. □

This proof shows clearly the connection between the condition on a single formal product in Theorem 4 and the condition on two products in Theorem 5. Unfortunately, we were unable to find a simple modification of this argument that works for the case $\star = \top$, mostly due to the issue presented in Remark 23.

Now we address the proof of Theorem 6. For this, we need the following result, which is an extension of Lemma 3 in [9], and whose proof is straightforward using similar arguments to the ones in [9].

Lemma 24. *Let $\mathcal{S}$ be a subset of $\mathbb{C}\cup\{\infty\}$ and let $p \in \mathbb{N}$. Then the set*

$$\sqrt[p]{\mathcal{S}} := \{z \in \mathbb{C}\cup\{\infty\} \mid z^p \in \mathcal{S}\}$$

is $\star$-reciprocal free if and only if $\mathcal{S}$ is $\star$-reciprocal free. Moreover, $\mathcal{S}$ is $\star$-reciprocal free if and only if $-\mathcal{S} := \{z \in \mathbb{C}\cup\{\infty\} \mid -z \in \mathcal{S}\}$ is $\star$-reciprocal free. (We use $\infty^p = \infty$ and $-\infty = \infty$.)

In view of the reduction presented in Section 5.1, we can assume in what follows that the matrices A_k, C_k are upper triangular and B_k, D_k are lower triangular, for $k = 1, \dots, r$.

We will use the following generalization of Lemma 21.

Lemma 25. *Let the matrix N be defined as follows*

$$N = \begin{bmatrix} A_1 & B_1 & & \\ & \ddots & \ddots & \\ & & \ddots & B_{s-1} \\ B_s & & & A_s \end{bmatrix}$$

where A_k and B_k are $n \times n$ upper triangular matrices, for $k = 1, \dots, s$. Then

$$\det N = \prod_{i=1}^{n} \left(\prod_{k=1}^{s} (A_k)_{ii} - (-1)^s \prod_{k=1}^{s} (B_k)_{ii} \right).$$

Proof. Consider the commutation matrix $P_{n,s}$ (as defined in Section 5.2.1), i.e., the permutation matrix corresponding to the following ordering of $\{1, \dots, sn\}$

$$(1, n+1, 2n+1, \dots, (s-1)n+1, 2, n+2, \dots, (s-1)n+2, \dots, n, 2n, \dots, sn).$$

The matrix $P_{n,s}^\top N P_{n,s}$ is block upper triangular with n diagonal blocks of size s of the type

$$\begin{bmatrix} (A_1)_{ii} & (B_1)_{ii} & & \\ & (A_2)_{ii} & \ddots & \\ & & \ddots & (B_{s-1})_{ii} \\ (B_s)_{ii} & & & (A_s)_{ii} \end{bmatrix}, \qquad i = 1, \dots, n.$$

Since $\det(P_{n,s}^\top N P_{n,s}) = \det N$, the result follows applying Lemma 21 to the diagonal blocks of $P_{n,s}^\top N P_{n,s}$. □

Proof of Theorem 6. Let us consider a periodic Schur decomposition of the matrices $A_1, \dots, A_r, B_1^\star, \dots, B_r^\star$ and $C_1, \dots, C_r, D_1^\star, \dots, D_r^\star$, namely:

$$\begin{aligned} Q_k^* A_k Z_k &= \widetilde{A}_k, & Q_{r+k}^* B_k^\star Z_{r+k} &= \widetilde{B}_k^\star, & k &= 1, \dots, r, \\ Q_k^* C_k Z_{k+1} &= \widetilde{C}_k, & Q_{r+k}^* D_k^\star Z_{r+k+1} &= \widetilde{D}_k^\star, & k &= 1, \dots, r, \end{aligned}$$

with $Z_{2r+1} := Z_1$, and $\widetilde{A}_k, \widetilde{B}_k^\star, \widetilde{C}_k, \widetilde{D}_k^\star$ being upper triangular, for all $k = 1, \dots, r$. Multiplying the pencil $\mathcal{Q}(\lambda)$ in (10) on the left and on the right by, respectively, $\operatorname{diag}(Q_1^*, \dots, Q_{2r}^*)$ and $\operatorname{diag}(Z_1, \dots, Z_{2r})$, we arrive at a pencil $\widetilde{\mathcal{Q}}(\lambda)$ which is strictly equivalent to $\mathcal{Q}(\lambda)$ and whose blocks are all upper triangular. Then $\mathcal{Q}$ is regular if and only if $\widetilde{\mathcal{Q}}$ is regular and, when the pencils are regular, we have $\Lambda(\mathcal{Q}) = \Lambda(\widetilde{\mathcal{Q}})$. So we can restrict our analysis to the case where all matrices $A_k, B_k^\star, C_k$, and $D_k^\star$ are upper triangular.

Applying Lemma 25 we arrive at

$$\det \mathcal{Q}(\lambda) = \prod_{i=1}^{n} \left(\lambda^{2r} \prod_{k=1}^{r} (A_k)_{ii} (B_k^{\star})_{ii} + \prod_{k=1}^{r} (C_k)_{ii} (D_k^{\star})_{ii} \right).$$

Therefore, $\mathcal{Q}(\lambda)$ is regular if and only if, for each $i = 1, \ldots, n$, at least one between $\prod_{k=1}^{r}(A_k)_{ii}(B_k^{\star})_{ii}$ and $\prod_{k=1}^{r}(C_k)_{ii}(D_k^{\star})_{ii}$ is nonzero.

Moreover, when $\mathcal{Q}(\lambda)$ is regular, its spectrum is $\Lambda(\mathcal{Q}) = \sqrt[2r]{\mathcal{S}}$, where

$$\mathcal{S} := \left\{ -\prod_{k=1}^{r} \frac{(C_k)_{ii}(D_k^{\star})_{ii}}{(A_k)_{ii}(B_k^{\star})_{ii}} \;,\quad i = 1, \ldots, n \right\}.$$

Notice that the numbers in $\mathcal{S}$ are minus the inverses of the eigenvalues of the formal product (9).

Let us first assume that (4) is nonsingular. Then, by Theorem 5 and the previous reasonings, $\mathcal{Q}(\lambda)$ is regular. Part (i) in the statement (the case $\star = *$) is a consequence of Lemma 24. Part (ii) (the case $\star = \top$) is a consequence of the same lemma and the identities

$$\sqrt[2r]{\mathcal{S}} \setminus \mathfrak{R}_{2r} = \sqrt[2r]{\mathcal{S} \setminus \{1\}}, \quad \text{and} \quad -(\mathcal{S} \setminus \{1\}) = -\mathcal{S} \setminus \{-1\}, \qquad (27)$$

valid for any $\mathcal{S} \subseteq \mathbb{C} \cup \{\infty\}$. More precisely, by Theorem 5, $\Lambda(\Pi)^{-1} \setminus \{-1\} = -\mathcal{S} \setminus \{-1\}$ is reciprocal free, where $\Lambda(\Pi)^{-1} := \{\lambda^{-1} : \lambda \in \Lambda(\Pi)\}$. Now, the second identity in (27), together with Lemma 24, imply that $\mathcal{S} \setminus \{1\}$ is reciprocal free as well, and the same lemma, together with the first identity in (27), imply that $\sqrt[2r]{\mathcal{S} \setminus \{1\}}$ is reciprocal free. Moreover, since the multiplicity of -1 as an eigenvalue of the formal product Π is at most one, we have that the multiplicity of each eigenvalue of $\mathcal{Q}(\lambda)$ belonging to $\mathfrak{R}_{2r}$ is at most one.

Conversely, let us assume that $\mathcal{Q}(\lambda)$ is regular and either (i) or (ii) in the statement of the theorem is satisfied. Then Lemma 24 implies that, if $\star = *$, the spectrum of the formal product (9) is $*$-reciprocal free. However, if $\star = \top$, Lemma 24 together with (27) imply that $\Lambda(\Pi) \setminus \{-1\}$ is reciprocal free, with Π as in (9), and $\lambda = -1$ has multiplicity at most 1. Then Theorem 5 implies that (4) is nonsingular. □

Remark 26. It is claimed in [14, p. 2], without proof, that the eigenvalues of the pencil

$$\begin{bmatrix} 0 & & & F_p \\ F_1 & \ddots & & \\ & \ddots & \ddots & \\ & & F_{p-1} & 0 \end{bmatrix} - \lambda \begin{bmatrix} E_p & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_{p-1} \end{bmatrix} \qquad (28)$$

are the pth roots of the eigenvalues of the formal product $E_p^{-1}F_pE_{p-1}^{-1}F_{p-1}\cdots E_1^{-1}F_1$, with E_k, F_k being $n \times n$ matrices, for $k = 1, \ldots, p$. Using this fact, Theorem

6 would follow immediately from Theorem 5. Indeed, after multiplying on the left and on the right the pencil (10) by the block antidiagonal matrix

$$R := \begin{bmatrix} & & I \\ & \iddots & \\ I & & \end{bmatrix}$$

we arrive at a pencil like (28) with $p = 2r$ and $E_1 = -B^\star_{r-1}, \ldots, E_{r-1} = -B^\star_1, E_r = -A_r, E_{r+1} = -A_{r-1}, \ldots, E_{p-1} = -A_1, E_{2r} = -B^\star_r, F_1 = D^\star_{r-1}, \ldots, F_{r-1} = D^\star_1, F_r = C_r, F_{r+1} = C_{r-1}, \ldots, F_{2r-1} = C_1, F_{2r} = -D^\star_r$.

7 An $O(n^3 r)$ algorithm for computing the solution

Here we describe an efficient algorithm for the solution of a nonsingular system of r Sylvester-like equations (3) of size $n \times n$. We follow the big-oh notation $O(\cdot)$, as in [16], for both large and small quantities, and we use the number of floating point operations (flops) as a complexity measure.

The tools needed to develop the algorithm are the same used, in the previous sections, for the nonsingularity results. In the description of the algorithm we focus on the complex case and so we consider triangular coefficients. However, a solution with quasitriangular forms in case of real data can be done following a similar procedure.

We proceed through the following steps:

1. (Step 1) We perform a suitable number of substitutions, changes and elimination of variables, in order to transform the system into irreducible systems of periodic form (4), as described in Section 4.

2. (Step 2) For each (irreducible) periodic system, we compute a periodic Schur decomposition to reduce the coefficients, say A_k, B_k, C_k, D_k, to upper and lower triangular forms, as described in Section 5.1.

3. (Step 3) Since the resulting systems can be seen as essentially block triangular linear systems (as described in Section 5.2.1), we solve them by back substitution.

4. (Step 4) We compute the value of the variables that have been eliminated in Step 1 (using Theorem 11).

This section describes how to handle these steps algorithmically. Moreover, we perform an analysis of the computational costs, showing that the solution can be computed in $O(n^3 r)$ time, and we prove a backward stability result for the computed solution.

Concerning Step 1, the discussions at the end of Sections 4.1 and 4.2, together with Lemma 12 provide an effective reduction process. Similarly, Step 2 can be solved by the computation of a periodic Schur factorization, which can be carried out in $O(n^3 r)$ flops. We refer to [3] for details concerning this reduction step. Step 4 amounts to applying formula (11) several times.

It remains to study how to effectively solve any of the "large" systems obtained by applying the re-ordered vec operator (Step 3). We focus on the case $s = \star$, since the case $s = 1$ can be found in [5]. The cases $\star = \top$ and $\star = *$ are handled in a similar way, but the first one is easier to describe since the associated system is linear, without the need of separating the real and imaginary parts. We will accurately describe the procedure for the solution of the former problem, and briefly explain the extensions needed to handle the latter case.

7.1 Solving the triangular system

We sort the variables in the triangular system (16), re-ordering the equations and unknowns by means of any of the echelon-shaped orderings described in Section 5.2.1, with the constraint that $(i, j, k) \prec (i, j, k')$ when $k < k'$. Two examples of such echelon-shaped orderings are $\preceq_A$ and $\preceq_{RC}$. The specific choice of the ordering has no effect on the computational cost, but might make it easier to parallelize the algorithm (see Remark 29).

The reordered system is block upper triangular with $\frac{n(n+1)}{2}$ diagonal blocks of order r and $2r$ (one for each non-ordered pair $\{i, j\}$, including the cases where $i = j$). We refer to these as the *small systems* $\mathbb{S}_{ij}$.

We provide in this section a high-level overview of the solution of this system by block back substitution, and in Sections 7.2 and 7.3 we describe how to perform it within the required computational cost.

At each of the $\frac{n(n+1)}{2}$ steps of the back substitution process, we need to solve a square linear system of the form:

$$M_{ij}\mathcal{X}_{ij} = \mathcal{E}_{ij} - \mathcal{F}_{ij}, \tag{29}$$

where M_{ij} is defined in (17) (when $i = j$) and (21) (when $i \neq j$); the vector $\mathcal{X}_{ij}$ has r (if $i = j$) or $2r$ (if $i \neq j$) components, obtained by stacking vertically all the entry unknowns $(X_1)_{ii}, \dots, (X_r)_{ii}$ (when $i = j$) or $(X_1)_{ij}, \dots, (X_r)_{ij}$ followed by $(X_1)_{ji}, \dots, (X_r)_{ji}$ (when $i \neq j$); the vector $\mathcal{F}_{ij}$ is defined as

$$\mathcal{F}_{ij} := \begin{cases} w_{ii} & \text{if } i = j, \\ \left[\begin{smallmatrix} w_{ij} \\ w_{ji} \end{smallmatrix}\right] & \text{otherwise,} \end{cases}$$

where w_{ij} is given by

$$w_{ij} := \begin{bmatrix} v_{ij1} \\ \vdots \\ v_{ijr} \end{bmatrix}, \quad v_{ijk} := \sum_{\substack{s \geq i, t \geq j \\ (s,t) \neq (i,j)}} \left((A_k)_{is}(X_k)_{st}(B_k)_{tj} - (C_k)_{is}(X_{k+1})_{st}(D_k)_{tj} \right); \tag{30}$$

and $\mathcal{E}_{ij}$ contains all the entries in position (i, j) (when $i = j$) or (i, j) and (j, i) (when $i \neq j$) of $E_1, \dots, E_r$ stacked vertically, according to the order in $\mathcal{F}_{ij}$. We identify X_{r+1} with $X_1^\star$ for simplicity.

Note that the values of the unknowns appearing in $\mathcal{F}_{ij}$ have been already computed if the linear systems are solved in decreasing order with respect to an echelon-shaped ordering (such as $\preceq_A$ or $\preceq_{RC}$).

The case $\star = *$ can be handled in a similar way, even if the associated system $\mathbb{S}$ is nonlinear. In Section 5.3, we have seen how the system can be linearized over $\mathbb{C}$ by doubling the number of equations. Here we follow a different approach: we consider it as a larger linear system over $\mathbb{R}$ of double the dimension in the variables $\mathrm{re}(\mathcal{X}_{ij})$ and $\mathrm{im}(\mathcal{X}_{ij})$. More precisely, the system $\mathbb{S}_{ii}$, when $\star = *$, is equivalent to the linear system over $\mathbb{R}$ defined by

$$\begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{r-1} \\ \beta_r & & & \alpha_r \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_{r-1} \\ Z_r \end{bmatrix} = \begin{bmatrix} U_1 \\ \vdots \\ U_{r-1} \\ U_r \end{bmatrix}, \qquad \begin{cases} Z_k &= \begin{bmatrix} \mathrm{re}(X_k)_{ii} \\ \mathrm{im}(X_k)_{ii} \end{bmatrix}, \\ U_k &= \begin{bmatrix} \mathrm{re}((E_k)_{ii} - (v_{ii})_k) \\ \mathrm{im}((E_k)_{ii} - (v_{ii})_k) \end{bmatrix}, \end{cases}$$

where α_k, β_k are 2×2 matrices defined, respectively, by

$$\begin{bmatrix} \mathrm{re}((A_k)_{ii}(B_k)_{ii}) & -\mathrm{im}((A_k)_{ii}(B_k)_{ii}) \\ \mathrm{im}((A_k)_{ii}(B_k)_{ii}) & \mathrm{re}((A_k)_{ii}(B_k)_{ii}) \end{bmatrix}, \quad \begin{bmatrix} \mathrm{re}((C_k)_{ii}(D_k)_{ii}) & -\mathrm{im}((C_k)_{ii}(D_k)_{ii}) \\ \mathrm{im}((C_k)_{ii}(D_k)_{ii}) & \mathrm{re}((C_k)_{ii}(D_k)_{ii}) \end{bmatrix},$$

when $k < r$, and by

$$\begin{bmatrix} \mathrm{re}((A_r)_{ii}(B_r)_{ii}) & -\mathrm{im}((A_r)_{ii}(B_r)_{ii}) \\ \mathrm{im}((A_r)_{ii}(B_r)_{ii}) & \mathrm{re}((A_r)_{ii}(B_r)_{ii}) \end{bmatrix}, \quad \begin{bmatrix} \mathrm{re}((C_r)_{ii}(D_r)_{ii}) & \mathrm{im}((C_r)_{ii}(D_r)_{ii}) \\ \mathrm{im}((C_r)_{ii}(D_r)_{ii}) & -\mathrm{re}((C_r)_{ii}(D_r)_{ii}) \end{bmatrix},$$

when $k = r$. Notice that the only differences between the two cases are the signs in the matrix on the right; this is due to the conjugation appearing in the last equation. The systems obtained for $\mathbb{S}_{ij}$ are defined similarly.

We will show, in Section 7.2, that the components v_{ijk} can be computed recursively so that, for each (i, j), the computation of $\mathcal{F}_{ij}$ requires only $O(nr)$ flops.

Moreover, we will show, in Section 7.3, that the system $M_{ij}\mathcal{X}_{ij} = \mathcal{E}_{ij} - \mathcal{F}_{ij}$, once the right-hand side term has been computed, can be solved in linear time, that is in $O(r)$ flops, thanks to the special structure of the matrix M_{ij}.

With all the above tools we can formulate Algorithm 1 to compute the solution of a periodic system of r generalized Sylvester equations whose coefficients are in upper and lower triangular form as in Section 5.1. Besides the computation of the solution X_k, the routine also computes the matrices $X_k B_k$ and $X_k D_k$, here denoted X_k^B and X_k^D, respectively, which are needed for an efficient computation of the right-hand side $\mathcal{E}_{ij} - \mathcal{F}_{ij}$ of the linear system.

Section 7.2 is devoted to describe the routine ComputeF, that computes the term $\mathcal{F}_{ij}$ in the right-hand side of the systems $\mathbb{S}_{ij}$, while Section 7.3 describes the solution of the system, that is the routine SolveIntermediateSystem. An algorithmic description of the former is given in Algorithm 2, while the latter procedure is outlined in algorithmic form in the proof of Lemma 28. A FORTRAN implementation of the code is available at `https://github.com/numpi/starsylv/`.

Algorithm 1 Solution of a periodic system of generalized $\star$-Sylvester equations

procedure GENERALIZEDSTARSYLVESTERSYSTEM$(A_k, B_k, C_k, D_k, E_k)$
  for $k = 1, \ldots, r$ **do**
    $X_k \leftarrow 0_{n\times n}$ ▷ we store the solution here
    $X_k^B \leftarrow 0_{n\times n}$ ▷ storage for X_k^B
    $X_k^D \leftarrow 0_{n\times n}$ ▷ storage for X_k^D
  end for
  for $(i, j) \in \{1, 2, \ldots, n\}^2$ with $i \le j$, decreasingly ordered by $\preceq_S$ **do**
    $\mathcal{F}_{ij} \leftarrow$ COMPUTEF$(X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j)$
    $x \leftarrow$ SOLVEINTERMEDIATESYSTEM$(M_{ij}, \mathcal{E}_{ij} - \mathcal{F}_{ij})$
    for $k = 1, \ldots, r$ **do**
      $[X_k]_{ij} \leftarrow x_k$
      $[X_k^B]_{ij} \leftarrow (e_i^\top X_k)(B_k e_j)$
      $[X_k^D]_{ij} \leftarrow (e_i^\top X_{k+1})(D_k e_j)$ ▷ with the convention $X_{r+1} = X_1^\star$
    end for
  end for
  return X_k
end procedure

7.2 Computing the term $\mathcal{F}_{ij}$

The computation of the term $\mathcal{F}_{ij}$, if evaluated directly using Equation (30), requires $O(n^2 r)$ multiplications and additions. However, by reusing some intermediate quantities computed in the previous steps, the computation can be carried out in $O(nr)$ flops.

Assume that $\mathcal{F}_{i'j'}$ have been computed for (i', j') with $i' \ge i$ and $j' \ge j$ and $(i', j') \ne (i, j)$, and for (j', i') with $j' \ge j$ and $i' \ge i$ and $(j', i') \ne (j, i)$, according to an echelon-shaped ordering. To evaluate $\mathcal{F}_{ij}$, we rearrange the first term in the definition of v_{ijk} (and similarly for v_{jik}) as follows:

$$\sum_{\substack{s\ge i, t\ge j\\ (s,t)\ne(i,j)}} (A_k)_{is}(X_k)_{st}(B_k)_{tj} = \sum_{t>j} (A_k)_{ii}(X_k)_{it}(B_k)_{tj} + \sum_{s>i, t\ge j} (A_k)_{is}(X_k)_{st}(B_k)_{tj}.$$

The first summand in the right-hand side of the above equation can be computed in $O(n)$ flops for a given k, so we only need to deal with the efficient evaluation of the latter summand. We can re-arrange it as follows:

$$\sum_{s>i, t\ge j} (A_k)_{is}(X_k)_{st}(B_k)_{tj} = \sum_{s>i} (A_k)_{is} \underbrace{\sum_{t\ge j} (X_k)_{st}(B_k)_{tj}}_{=:(X_k^B)_{sj}} =: \sum_{s>i} (A_k)_{is}(X_k^B)_{sj},$$

and this can be computed in $O(n)$ flops if $(X_k^B)_{sj}$, for $s > i$, is known. The idea is to compute and store $(X_k^B)_{ij}$, after the computation of $(X_k)_{ij}$, for every i

and j, and use it in the following steps. Notice that the computation of $(X_k^B)_{ij}$ requires only $O(n)$ operations since $(X_k^B)_{ij}$ is the element in position (i,j) of the product $X_k B_k$, and can be computed immediately after $(X_k)_{ii}$; in fact it depends only on entries of X_k that are known, thanks to the triangular structure of B_k.

As described in Algorithm 1, $(X_k^B)_{sj}$ has been precomputed in the previous steps, after the computation of $(X_k)_{sj}$. Thus, we can evaluate the first addend of v_{ijk} by computing a summation of $O(n)$ elements, so by means of $O(n)$ flops.

Using as above the notation $X_{r+1} := X_1^\star$, a similar formula holds for the second term, which can be written as

$$\sum_{\substack{s\geq i, t\geq j\\(s,t)\neq(i,j)}} (C_k)_{is}(X_{k+1})_{st}(D_k)_{tj} = \sum_{t>j}(C_k)_{ii}(X_{k+1})_{it}(D_k)_{tj} + \sum_{s>i}(C_k)_{is}\underbrace{\sum_{t>j}(X_{k+1})_{st}(D_k)_{tj}}_{:=(X_k^D)_{sj}},$$

and can be computed in $O(n)$ by storing the computed $(X_k^D)_{sj}$ at every step, as we have done with $(X_k^B)_{sj}$.

An algorithmic description of the above process, which can be plugged in directly in Algorithm 1, is given in Algorithm 2, and clearly requires $O(nr)$ arithmetic operations. Notice that in the pseudocode of Algorithm 2 all scalar products are computed on the complete rows and columns of the matrices $X_1,\dots,X_r$. This is done just for notational convenience but the formulation of Algorithm 2 is equivalent to formula (30), thanks to the initialization to zero of X_k, X_k^B, and X_k^D, for $k=1,\dots,r$. Nevertheless, in the implementation it is convenient to skip all the entries that are known to be zero.

Remark 27. In Algorithm 1 we have shown that it is possible to compute $(X_k^B)_{ij}$ and $(X_k^D)_{ij}$ after the solution of the linear system. In fact, a careful look at the algorithm shows that the scalar products

$$[X_k^B]_{ij} \leftarrow (e_i^T X_k)(B_k e_j), \qquad [X_k^D]_{ij} \leftarrow (e_i^T X_{k+1})(D_k e_j)$$

can be avoided. All non-zero elements in the above summations, except the ones corresponding to the diagonal entries of X_k and B_k or D_k, are already computed and summed up in ComputeF. Thus, the entries in position (i,j) of X_k^B and X_k^D can be computed with an $O(1)$ update of these partial sums. This does not change the asymptotic cost, but slightly improves the timing and it has been exploited in the implementation. However, we decided to avoid describing it in detail in the pseudocode for the sake of simplicity.

7.3 Solving the small linear systems

We describe how to efficiently solve the linear system (29) involving the matrix M_{ij}. The two cases $i=j$ and $i\neq j$ are different in the dimension of the matrix,

Algorithm 2 Subroutines used to compute the entries of $\mathcal{F}_{ij}$, which is part of the right-hand side of the linear system.

procedure ComputeF($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
  if $i = j$ **then**
    $F \leftarrow$ ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
  else
    $F(1:r) \leftarrow$ ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
    $F(r+1:2r) \leftarrow$ ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, j, i$)
  end if
  return F
end procedure
procedure ComputeW($X_k, X_k^B, X_k^D, A_k, B_k, C_k, D_k, i, j$)
  $F \leftarrow 0_r$
  for $k = 1, \ldots, r$ **do**
    $f_1 \leftarrow (A_k)_{ii}(e_i^\top X_k)(B_k e_j) + (e_i^\top A_k)(X_k^B e_j)$
    $f_2 \leftarrow (C_k)_{ii}(e_i^\top X_{k+1})(D_k e_j) + (e_i^\top C_k)(X_k^D e_j)$ ▷ With $X_{r+1} = X_1^\star$
    $F_k \leftarrow f_1 + f_2$
  end for
  return F
end procedure

but they share the same structure, so we can handle them at the same time. More precisely, we have the following result for the $\star = \top$ case.

Lemma 28. *Let M be an $\ell \times \ell$ matrix such that the elements in position (i, j) are allowed to be nonzero only if $0 \leq j - i \leq 1$ or if $(i, j) = (\ell, 1)$. Then M admits a QR factorization $M = QR$ where R is upper bidiagonal except in the last column, and Q is a product of $\ell - 1$ plane rotations.*

Proof. The proof is constructive and by induction. The case $\ell = 1$ is trivial, so let us assume that we have an $(\ell+1)\times(\ell+1)$ matrix M, so that we can compute a rotation G acting on the first and last row that annihilates the elements in position $(\ell + 1, 1)$. More precisely

$$GM = G \begin{bmatrix} \times & \times & & \\ & \ddots & \ddots & \\ & & \times & \times \\ \times & & & \times \end{bmatrix} = \left[\begin{array}{c|c} a_1 & b_1 \qquad x_1 \\ \hline & \widetilde{M} \end{array}\right],$$

where $\widetilde{M}$ has the same shape as M, but is of size $\ell\times\ell$. Therefore, we can factorize $\widetilde{M} = \widetilde{Q}\widetilde{R}$, with $\widetilde{Q}$ being the product of $\ell - 1$ rotations. Setting $Q := G^\star \left[\begin{smallmatrix} 1 & 0 \\ 0 & Q \end{smallmatrix}\right]$ and

$$R = \left[\begin{array}{c|c} a_1 & b_1 \qquad x_1 \\ \hline & \widetilde{R} \end{array}\right]$$

concludes the proof. □

The above proof shows that the matrices Q and R can be computed in $O(\ell)$, and then the linear system $Mx = QRx = y$ can be solved in $O(\ell)$ by the application of $O(\ell)$ rotations to y (each of these operations can be done in $O(1)$) and by a back substitution, that, thanks to the sparsity of R, can be computed in $O(\ell)$ as well.

In our case the matrix of the linear system has $\ell \in \{r, 2r\}$, so we can solve each intermediate linear system in $O(r)$.

The case $\star = *$ is not much different, since the matrices M_{ij} of the linear system are block bidiagonal (except for the block at the end of the first column), with 2×2 blocks. In fact, the matrices M_{ij} can be brought into upper triangular form using about $5r$ rotations, and the upper triangular form enjoys a block bidiagonal form that allows us to solve the linear system in $O(r)$.

Lemma 28 can be easily converted into a routine and provides a possible implementation for SolveIntermediateSystem in Algorithm 1. An implementation for this routine can be found in the code used for the tests, available at `https://github.com/numpi/starsylv/`.

7.4 Computational cost and storage

We evaluate the total computational cost of the algorithm (in terms of floating-point operations) by taking into account the cost of all single steps.

Step 1 requires only some bookkeeping and possibly swapping and transposing matrices in memory, but no floating point operations. This step produces several periodic systems; let $r_1, r_2, \ldots, r_m$ be their sizes, with $r_1 + \cdots + r_m \le r$. We prove that each of these systems is solved using $O(n^3 r_i)$ flops.

Step 2 (for the ith periodic system of size r_i) requires computing a periodic Schur form, which costs $O(n^3 r_i)$ with the algorithm of [3]. Once the periodic Schur form has been computed, the changes of variables amount to $O(r_i)$ products between $n \times n$ matrices.

In Step 3, the method described in Section 7.2 allows one to compute each of the $\frac{n(n+1)}{2}$ terms $\mathcal{F}_{ij}$ in $O(nr_i)$ time, and Section 7.3 shows how to solve in $O(r_i)$ time the linear systems required in each of the $\frac{n(n+1)}{2}$ back substitution steps. The total amount of flops required by this step is, thus, $O(n^3 r_i)$.

Step 4 requires applying formula (11) (which costs $O(n^3)$ to compute) once for each remaining variable, that is, at most $r-1$ times.

Combining all the above steps we obtain an algorithm with a total cost of $O(n^3 r)$ flops. Moreover, the only storage required during the operation is the one of $O(r)$ matrices of size $n\times n$, so the storage required is $O(n^2 r)$, which is optimal (given that the same amount of storage is required to store the solutions).

Remark 29. If the anti-diagonal ordering $\preceq_A$ is used, then the entries on each anti-diagonal ($i + j =$ constant) can be computed independently in a parallel fashion, since they do not depend on each other. See Figure 3 for an example.

$$\begin{bmatrix} 9 & 8 & 7 & 6 & 5 \\ 8 & 7 & 6 & 5 & 4 \\ 7 & 6 & 5 & 4 & 3 \\ 6 & 5 & 4 & 3 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

Figure 3: Ordering in which the elements need to be computed following the $\preceq_A$ ordering. The entries marked with the same number can be computed in parallel.

This suggests that, in principle, the algorithm could perform faster in a parallel implementation using this particular ordering.

Remark 30. Step 1 requires some discrete computations on the indices to identify the periodic systems and eliminate variables and equations; we have ignored them here since they involve no floating-point operations, but it can be shown that they can be performed in $O(r)$ operations with the help of a graph traversal algorithm.

7.5 Backward error analysis

Here we provide a backward error analysis of the algorithm described in the previous sections. We use the standard floating point number model with unit roundoff u and, for an expression ℓ, we denote by $\mathsf{fl}(\ell)$ the computed value of ℓ using floating point operations. We will use the notation

$$\gamma_k := \frac{cku}{1 - cku},$$

where c denotes a small constant, whose exact value is not relevant (see [16, p. 68]).

We assume, moreover, that all the linear systems $Ax = b$ that are encountered are solved using a backward stable method. More precisely, we say that an algorithm to solve a linear system $Ax = b$, with $A \in \mathbb{C}^{m\times m}$, has *backward error* ε_A if the computed solution $\widetilde{x} = \mathsf{fl}(A^{-1}b)$ is the exact solution of a perturbed system $(A + \delta A)\widetilde{x} = b$, with $\|\delta A\|_2/\|A\|_2 \le \varepsilon_A$. Note that only the coefficient matrix is perturbed (see [16, Th. 19.5] and the following discussion for an explanation). In the case of solving the system with the QR factorization using s Givens rotations, as we do in Section 7.3 with $s = O(r)$, this quantity can be taken as

$$\varepsilon_A = m \cdot \gamma_s$$

(see p. 368 and Theorem 19.10 in [16]). The factor m comes from the fact that the bound in [16] is only given column-wise and

$$\|\mathrm{Col}_j A\|_2 \le \|A\|_2 \le \sqrt{m}\|A\|_1 = \sqrt{m} \max_{j=1,\dots,m} \|\mathrm{Col}_j A\|_1 \le m \max_{j=1,\dots,m} \|\mathrm{Col}_j A\|_2, \tag{31}$$

for all $j = 1, \dots, m$, where $\mathrm{Col}_j A$ is the jth column of A (see, for instance, [16, Tables 6.1 and 6.2] for the last two inequalities).

We obtain a backward error result formulating the problem as a vectorized linear system. For simplicity, we will focus on periodic systems with upper and lower triangular coefficients in Theorem 31. The general case will be commented right after the proof.

Theorem 31. *Consider a system of equations of the form* (4)*, with* $A_k, C_k, B_k^\top, D_k^\top$ *being upper triangular, and let* $M\mathcal{X} = \mathcal{E}$ *be its vectorized form, where* $M \in \mathbb{C}^{rn^2 \times rn^2}$ *if* $\star = \top$*, or* $M \in \mathbb{R}^{2rn^2 \times 2rn^2}$ *if* $\star = *$.

When implemented in standard floating-point arithmetic, the algorithm described in Sections 7.1–7.3 *produces a result* $\widetilde{\mathcal{X}}$ *satisfying*

$$(M + \delta M)\widetilde{\mathcal{X}} = \mathcal{E} + \delta\mathcal{E}, \tag{32}$$

with $\|\delta M\|_2/\|M\|_2 \le r\,\gamma_r + \gamma_{n^2}(1 + r\,\gamma_r)$*,* $\|\delta\mathcal{E}\|_2/\|\mathcal{E}\|_2 \le \gamma_{n^2}$*.*

Remark 32. The reader may wonder if a stronger form of structured backward stability holds: the algorithm should produce matrices that satisfy

$$(A_k+\delta A_k)\widetilde{X}_{\alpha_k}^{s_k}(B_k+\delta B_k)-(C_k+\delta C_k)\widetilde{X}_{\beta_k}^{t_k}(D_k+\delta D_k) = E_k+\delta E_k \quad k = 1, \dots, r,$$

with $\|\delta S_k\|_2/\|S_k\|_2$ being small, for $S = A, B, C, D, E$. Unfortunately, algorithms of this family fail to be structurally backward stable even in the simplest case of a single Sylvester equation $AX - XD = E$, as shown in [15, §16.2] (see also the discussion in [5] for the case $s = 1$).

Note that Theorem 31 is nevertheless sufficient to show that the residual of each equation $R_k = \|A_k\widetilde{X}_{\alpha_k}^{s_k}B_k - C_k\widetilde{X}_{\beta_k}^{t_k}D_k - E_k\|_F$, for $k = 1, 2, \dots, r$, is small. Indeed, $\|M\widetilde{\mathcal{X}} - \mathcal{E}\|_2 = \sqrt{\sum_{k=1}^r R_k^2}$ satisfies

$$\frac{\|M\widetilde{\mathcal{X}} - \mathcal{E}\|_2}{\|M\|_2\|\widetilde{\mathcal{X}}\|_2 + \|\mathcal{E}\|_2} \le \max\left(\frac{\|\delta M\|_2}{\|M\|_2}, \frac{\|\delta\mathcal{E}\|_2}{\|\mathcal{E}\|_2}\right)$$

by [16, Thm 7.1].

In order to prove Theorem 31, we need the following technical results.

Lemma 33. *Let* $N \in \mathbb{C}^{m\times m}$ *and* $x, y \in \mathbb{C}^m$*, with* $x, y \neq 0$*, be such that*

$$y = (N + \Delta N)x, \qquad \frac{\|\Delta N\|_2}{\|N\|_2} \le \varepsilon, \tag{33}$$

for some $\varepsilon > 0$*. Let* $\delta y \in \mathbb{C}^m$ *be such that*

$$\frac{\|\delta y\|_2}{\|y\|_2} \le \kappa, \tag{34}$$

for some $\kappa > 0$*. Then*

$$y + \delta y = (N + \delta N)x,$$

for some $\delta N \in \mathbb{C}^{m\times m}$ *with*

$$\frac{\|\delta N\|_2}{\|N\|_2} \le \varepsilon + \kappa(1+\varepsilon).$$

Proof. From (33) and (34) we get

$$\|\delta y\|_2 \le \kappa\|y\|_2 \le \kappa\left(\|N\|_2 + \|\Delta N\|_2\right)\|x\|_2 \le \kappa(1+\varepsilon)\|N\|_2\|x\|_2. \qquad (35)$$

Now, setting $\widetilde{N} := \|x\|_2^{-2}\cdot(\delta y)x^*$, we have $\widetilde{N}x = \delta y$ and $\|\widetilde{N}\|_2 = \|\delta y\|_2/\|x\|_2$, so $\|\delta y\|_2 = \|\widetilde{N}\|_2\|x\|_2$. Then, by (35),

$$\|\widetilde{N}\|_2 \le \kappa(1+\varepsilon)\|N\|_2. \qquad (36)$$

Finally, taking $\delta N := \Delta N + \widetilde{N}$, and using (36), we arrive at

$$\|\delta N\|_2 \le \|\Delta N\|_2 + \|\widetilde{N}\|_2 \le (\varepsilon + \kappa(1+\varepsilon))\|N\|_2.$$

□

Lemma 34. *Consider a square linear system of the form*

$$Fx = b - \sum_{k=1}^{s} N_k c_k,$$

where $F, N_k \in \mathbb{C}^{m\times m}$, *and* $b, c_k \in \mathbb{C}^m$ *are given, for* $k = 1, \ldots, s$, *and* x *is the unknown.*

Forming the sum in the right-hand side, in floating point arithmetic, and then solving the linear system using an algorithm with backward error ε_F, *produces a computed solution* $\widetilde{x}$ *which is the exact solution of a perturbed system*

$$(F + \delta F)\widetilde{x} = b + \delta b - \sum_{k=1}^{s}(N_k + \delta N_k)c_k,$$

with

$$\frac{\|\delta F\|_2}{\|F\|_2} \le \varepsilon_F, \quad \frac{\|\delta b\|_2}{\|b\|_2} \le \gamma_s, \quad \frac{\|\delta N_k\|_2}{\|N_k\|_2} \le m\gamma_m + \gamma_s(1 + m\gamma_m).$$

Proof. Let $\widetilde{d}_k = \mathsf{fl}(N_k c_k)$, $\widetilde{f} = \mathsf{fl}(b - \sum_{k=1}^{s}\widetilde{d}_k)$. By hypothesis, $(F+\delta F)\widetilde{x} = \widetilde{f}$, with $\|\delta F\|_2/\|F\|_2 \le \varepsilon_F$. The usual backward error analysis of summation can be used to show that $\widetilde{f} = b + \delta b - \sum_{k=1}^{s}(\widetilde{d}_k + \delta\widetilde{d}_k)$, with $|(\delta b)_i|/|b_i|, |(\delta\widetilde{d}_k)_i|/|(\widetilde{d}_k)_i| \le \gamma_s$, for $i = 1, \ldots, m$ (see [16, Section 4]). Now, by standard backward error analysis of matrix-vector multiplication, we know that

$$\widetilde{d}_k = (N_k + \Delta N_k)c_k,$$

with $\|\mathrm{Col}_j(\Delta N_k)\|_2/\|\mathrm{Col}_j(N_k)\|_2 \leq \gamma_m$, for $j = 1, \ldots, m$ (see [16, Section 3.5]). Using (31), this implies $\|\Delta N_k\|_2/\|N_k\|_2 \leq m\gamma_m$. Now, we can apply Lemma 33, with $y = \widetilde{d}_k, \delta y = \delta\widetilde{d}_k$, $x = c_k$, $N = N_k$ and $\Delta N = \Delta N_k$, to conclude that

$$\widetilde{d}_k + \delta\widetilde{d}_k = (N_k + \delta N_k)c_k,$$

with $\|\delta N_k\|_2/\|N_k\|_2 \leq m\gamma_m + \gamma_s(1 + m\gamma_m)$, as wanted. □

Proof of Theorem 31. We note that each step of the block back substitution corresponds to solving a linear system of the form (29). More precisely, this system is

$$M_{ij}\mathcal{X}_{ij} = \mathcal{E}_{ij} - \sum_{(s,t)\in\mathcal{U}_{ij}} N_{st}^{(ij)}\mathcal{X}_{st},$$

where $\mathcal{U}_{ij} = \{(i', j') : \max\{i', j'\} \geq \max\{i, j\} \text{ and } \min\{i', j'\} \geq \min\{i, j\}\}$ and the matrices $N_{st}^{(ij)}$ are given by writing (30) in matrix form. By Lemma 34, there are some matrices δM_{ij} and $\delta N_{st}^{(ij)}$ such that

$$(M_{ij} + \delta M_{ij})\widetilde{\mathcal{X}}_{ij} = \mathcal{E}_{ij} + \delta\mathcal{E}_{ij} - \sum_{(s,t)\in\mathcal{U}_{ij}} (N_{st}^{(ij)} + \delta N_{st}^{(ij)})\widetilde{\mathcal{X}}_{st},$$

where $\widetilde{\mathcal{X}}_{ij}$ are the computed solutions at the (i, j) step and $\widetilde{\mathcal{X}}_{st}$, for $s \geq i, t \geq j$, with $(s, t) \neq (i, j)$, are the ones computed in the previous steps, and

$$\frac{\|\delta M_{ij}\|_2}{\|M_{ij}\|_2} \leq \varepsilon_{M_{ij}}, \quad \frac{\|\delta N_{st}^{(ij)}\|_2}{\|N_{st}^{(ij)}\|_2} \leq r\gamma_r + \gamma_{n^2}(1 + r\gamma_r), \quad \frac{\|\delta\mathcal{E}_{ij}\|_2}{\|\mathcal{E}_{ij}\|_2} \leq \gamma_{n^2}.$$

If the $r \times r$ (or $(2r) \times (2r)$) linear system is solved through the QR factorization of M_{ij}, then $\varepsilon_{M_{ij}} \leq r\gamma_r$, as mentioned before (see [16, Th. 19.10]).

This gives a backward error for each block-row of the matrix M and of the right-hand side $\mathcal{E}$ in Theorem 31. Since these rows are never reused between equations, this defines a perturbation of M and $\mathcal{E}$ which ensures (32). □

We note that Theorem 31 corresponds to Step 3 in the procedure described at the beginning of Section 7 for solving a general system (3). The remaining steps can be carried out also in a backward stable way, as we are going to explain.

Step 1 involves no computations, just relabeling of the equations, transpositions and conjugations (which are exact in floating point arithmetic).

Step 2 is backward stable since the periodic QZ algorithm relies on unitary transformations and the following change of variables is unitary.

In Step 4, the vectorization of (11) produces the linear system

$$(B_k^\top \otimes A_k)\,\mathrm{vec}(X_{\alpha_k}^{s_k}) = \mathrm{vec}(E_k) + (D_k^\top \otimes C_k)\,\mathrm{vec}(X_{\beta_k}^{t_k}),$$

which is again in the form treated in Lemma 34, so we only have to ensure that the method used to solve this linear system of the form $(B_k^\top \otimes A_k)\,\mathrm{vec}(X) =$

$\mathrm{vec}(F)$ is backward stable. To solve this system, we first compute $\widetilde{Y} = \mathsf{fl}(A_k^{-1}F)$ column by column, each time solving a linear system with A_k, and then similarly $\widetilde{X} = \mathsf{fl}(\widetilde{Y}B_k^{-1})$, solving a linear system for each of its rows.

We assume that the linear systems with A_k are solved with a backward stable method, i.e.,

$$(A_k + \delta_j A_k)\mathrm{Col}_j(\widetilde{Y}) = \mathrm{Col}_j(F), \quad \frac{\|\delta_j A_k\|_2}{\|A_k\|_2} \leq \varepsilon_{A_k},$$

(note that there is a different perturbation $\delta_j A_k$ for each j); hence we have

$$(\mathbb{A} + \delta\mathbb{A})\,\mathrm{vec}(\widetilde{Y}) = \mathrm{vec}(F), \quad \frac{\|\delta\mathbb{A}\|_2}{\|\mathbb{A}\|_2} \leq \varepsilon_{A_k},$$

where $\mathbb{A} = I_n \otimes A_k$ and $\delta\mathbb{A} = \mathrm{diag}(\delta_1 A_k, \ldots, \delta_n A_k)$.

An analogous argument shows that

$$(\mathbb{B} + \delta\mathbb{B})\,\mathrm{vec}(\widetilde{X}) = \mathrm{vec}(\widetilde{Y}), \quad \frac{\|\delta\mathbb{B}\|_2}{\|\mathbb{B}\|_2} \leq \varepsilon_{B_k},$$

where $\mathbb{B} = B_k^\top \otimes I_n$. Combining these two relations we have

$$\mathrm{vec}(F) = (\mathbb{A} + \delta\mathbb{A})(\mathbb{B} + \delta\mathbb{B})\,\mathrm{vec}(\widetilde{X}) = (\mathbb{A}\mathbb{B} + \delta(\mathbb{A}\mathbb{B}))\,\mathrm{vec}(\widetilde{X}),$$

with $\delta(\mathbb{A}\mathbb{B}) = \delta\mathbb{A}\cdot\mathbb{B} + \mathbb{A}\cdot\delta\mathbb{B} + \delta\mathbb{A}\cdot\delta\mathbb{B}$. Since $\|\mathbb{A}\mathbb{B}\|_2 = \|\mathbb{A}\|_2\|\mathbb{B}\|_2$ for our choice of $\mathbb{A}$ and $\mathbb{B}$ (thanks to the properties of the Kronecker product [17, p. 253]), we have the bound

$$\begin{aligned}\frac{\|\delta(\mathbb{A}\mathbb{B})\|_2}{\|\mathbb{A}\mathbb{B}\|_2} &= \frac{\|\delta\mathbb{A}\cdot\mathbb{B} + \mathbb{A}\cdot\delta\mathbb{B} + \delta\mathbb{A}\cdot\delta\mathbb{B}\|_2}{\|\mathbb{A}\|_2\|\mathbb{B}\|_2} \leq \frac{\|\delta\mathbb{A}\|_2\|\mathbb{B}\|_2 + \|\mathbb{A}\|_2\|\delta\mathbb{B}\|_2 + \|\delta\mathbb{A}\|_2\|\delta\mathbb{B}\|_2}{\|\mathbb{A}\|_2\|\mathbb{B}\|_2} \\ &\leq \varepsilon_{A_k} + \varepsilon_{B_k} + \varepsilon_{A_k}\varepsilon_{B_k}.\end{aligned}$$

As a consequence of these arguments, the procedure described at the beginning of Section 7 produces a backward stable algorithm for solving general systems of the form (3).

7.6 Numerical experiments

We have implemented the proposed algorithm for the solution in the case $\star = \top$. The case $\star = *$ can be obtained with minimal changes (from the algorithmic point of view), so we decided to avoid running the same experiments concerning stability and performance. We have run the tests on a server with a Xeon X5680 CPU and 24 GB of memory. Our implementation is available at `https://github.com/numpi/starsylv/`.

We have computed the CPU time required by our implementation as a function of the size of the matrices n and of the number of equations in the reduced system r, and we have compared it with the behavior predicted by our analysis.

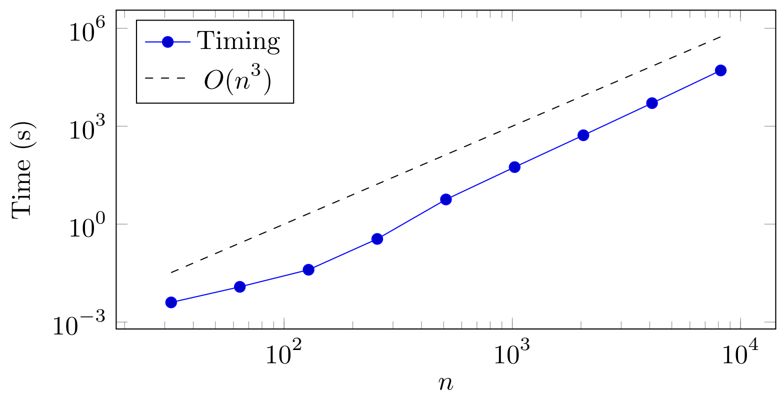

Figure 4: CPU time required by the algorithm described in Section 7.5 for the $\star = \top$ case, as a function of n. The timings reported are for a system with 3 equations, already in the required triangular form. The problems tested have sizes ranging from $n = 32$ to $n = 8192$.

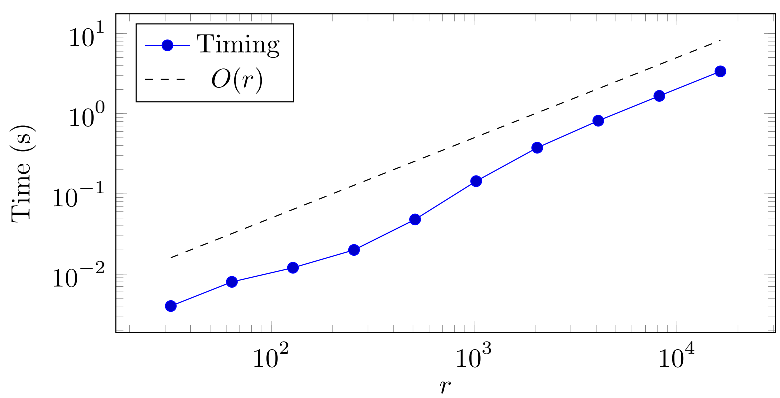

Figure 5: CPU time required by the algorithm described in Section 7.5 for the $\star = \top$ case, as a function of r. The timings reported are for a system with r equations and coefficient matrices of size 16×16, already in the required triangular form. The problems tested have sizes ranging from $r = 32$ to $r = 16384$.

We have considered only systems with triangular factors. The general case requires the reduction step to triangular factors, through the periodic Schur form as described in Section 5, that has been already implemented in [2].

The results are reported in Figure 4 for the CPU time required for the solution of a system of three equations with coefficients of variable size n, and in Figure 5 for a system of r equations of size 16. Both plots confirm the cubic and linear dependence of the CPU time on the parameters n and r, respectively, that we expect. The dashed lines in the two plots are obtained plotting the functions $k_n n^3$ and $k_r r$ for two appropriate constants k_n and k_r.

Beside timings, we have also tested the accuracy of the implementation. For each value of n and r we have generated several systems of $\top$-Sylvester equations (in the required triangular form), and we have computed the residuals $R_k := \|A_k X_k B_k - C_k X_{k+1} D_k - E_k\|_F$ for $k = 1, \dots, r-1$, and $R_r := \|A_r X_r B_r - C_r X_1^\top D_r - E_r\|_F$. Then, the 2-norm of the residual of the linear system can be evaluated as $R := \sqrt{R_1^2 + \cdots + R_r^2}$. In Figure 6 and Figure 7 we have plotted an upper bound of the relative residuals $R/\|M\|_2$, obtained using the relation $n\sqrt{r}\|M\|_2 \geq \|M\|_F$, where M is the matrix of the "large" linear system, for different values of n and r (recall that M has size $n^2 r$). Each value has been averaged over 100 runs. The Frobenius norm of M is easily computable recalling that, if two matrices M_1 and M_2 do not have non-zero entries in corresponding positions, then $\|M_1 + M_2\|_F^2 = \|M_1\|_F^2 + \|M_2\|_F^2$, and the relation $\|A \otimes B\|_F = \|A\|_F \|B\|_F$.

In these tests, the coefficients matrices A_k, B_k, C_k, D_k have been chosen with random entries with normal distribution, and with the correct triangular structure. We have then shifted A_k and B_k with $\sqrt{n}I$ to avoid finding solutions with very large norms.

From the tests performed so far, the algorithm behaves in a backward stable manner, as predicted by our analysis. In fact, one can spot that the error growth with respect to n and r is even less than the upper bound proved in this section. The error seems to grow slightly less than $\sqrt{n}$, and to be independent of r. This behavior is often encountered in dense linear algebra algorithms, since on average the errors do not accumulate in the same direction (see e.g. [16, Section 4.5]).

8 Conclusions and future work

We have provided necessary and sufficient conditions for the nonsingularity of r coupled generalized Sylvester and $\star$-Sylvester equations (3), with square coefficients of the same size $n \times n$. We have shown that the problem can be reduced to periodic systems having at most one generalized $\star$-Sylvester equation. A characterization for the nonsingularity of periodic systems of just generalized Sylvester equations was obtained in an unpublished work by Byers and Rhee [5]. That characterization was given in terms of spectral properties of matrix pencils constructed from the coefficient matrices of the system. We have provided

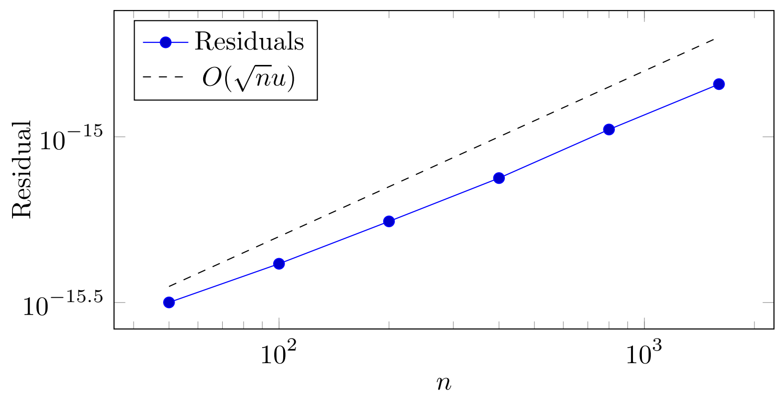

Figure 6: Average residuals of 100 systems of ⊤-Sylvester equations solved via the algorithm described in Section 7. The systems considered have 3 equations with a variable coefficient size n.

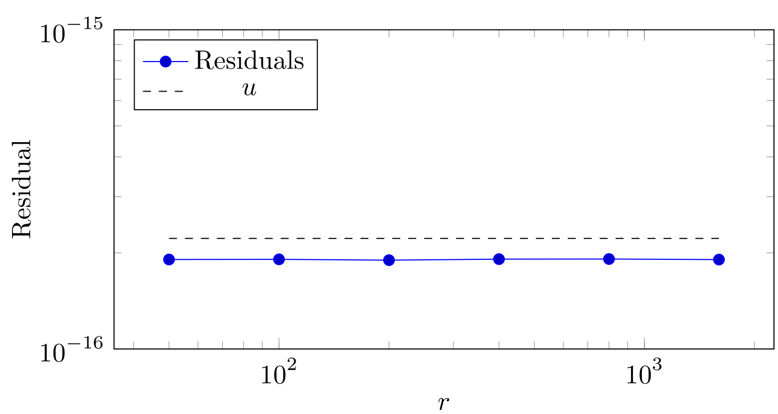

Figure 7: Average residuals of 100 systems of ⊤-Sylvester equations solved via the algorithm described in Section 7. The systems considered have coefficients with size 8×8, and r equations.

an analogous characterization for the nonsingularity of periodic systems with exactly one generalized $\star$-Sylvester equation. We have also provided a characterization for both types of periodic systems (namely, the one with exactly one generalized $\star$-Sylvester equation and the one with only generalized Sylvester equations) in terms of spectral properties of formal products constructed from the coefficient matrices of the system. We have also presented an $O(n^3r)$ algorithm for computing the unique solution of a nonsingular system, which has been shown to be backward stable.

A future research line that naturally arises from this work is to get a characterization of nonsingularity in the more general setting of rectangular coefficient matrices.

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