Uniqueness of solution of generalized Sylvester-like equations with rectangular coefficients*

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Abstract

We provide necessary and sufficient conditions for the uniqueness of solution of the generalized Sylvester and $\star$-Sylvester equations, $AXB - CXD = E$ and $AXB + CX^\star D = E$, respectively, where the matrix coefficients are allowed to be rectangular. These conditions generalize existing results for the same equations with square coefficients. Our characterization reveals unexpected constraints regarding the size and invertibility of some of the coefficient matrices, in addition to the same spectral conditions that appear in the square case.

Keywords. Sylvester equation, eigenvalues, matrix pencil, matrix equation

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1 Introduction

We consider the *generalized Sylvester equation*

$$AXB - CXD = E, \tag{1}$$

and the *generalized $\star$-Sylvester equation*

$$AXB + CX^\star D = E \tag{2}$$

for the unknown $X \in \mathbb{C}^{m\times n}$, with $\star$ being either the transpose ($\top$) or the conjugate transpose ($*$), and A, B, C, D, E being matrices with appropriate size. We are interested in the most general situation, where both the coefficients and the unknown are allowed to be rectangular.

Sylvester-like equations are among the most popular matrix equations, and they arise in many applications (see, for instance, [1–3, 11] and the review [14]). In particular, equations with rectangular coefficients arise in several eigenvalue perturbation and updating problems [10, 15].

Two of the most relevant questions when dealing with linear equations are: (a) when does the equation have a solution (*consistency*)?, and (b) when does the equation have exactly one solution (*uniqueness*)?

It is trivial to obtain solutions to these problems in terms of the matrices obtained by vectorization of the equations; for instance, Equation (1) has a unique solution if and only if the matrix $B^\top \otimes A - D^\top \otimes C$ is invertible (see e.g. [9, Section 4.3]). In many cases, however, it is possible to find conditions of a different kind, related to the spectral properties of smaller matrices and pencils: for instance, in the case of square coefficients it is proved in [4, Theorem 1] that the solution of (1) is unique if and only if the matrix pencils $A - \lambda C$ and $D - \lambda B$ are regular and do not have common eigenvalues. Again for square coefficients, it is proved in [6, Th. 15] that the uniqueness of solution of (2) can be characterized in terms of elementary spectral properties of the matrix pencil

$$\mathcal{Q}(\lambda) = \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \tag{3}$$

(see Theorem 4 for a precise statement).

This kind of characterizations are often easier to use, both in theory and in numerical computations, than the ones based on Kronecker products; the main reason is that they involve matrices or matrix pencils with smaller size, of the same order as the coefficients. For instance, in the case where all matrices are $n \times n$ they are given in terms of matrix pencils with size n or $2n$ instead of matrices with size n^2.

Several authors have given conditions of this kind. A condition for consistency for both (1) and (2) has been recently given in [7] in the most general situation, where both the coefficients and the unknown have arbitrary size (square or rectangular). As for the uniqueness, a characterization for Equation (1) was obtained in [4] in the case where $A, C \in \mathbb{C}^{m\times m}$ and $B, D \in \mathbb{C}^{n\times n}$ (so the coefficients are all square but not necessarily with the same size). A condition for the uniqueness of solution of Equation (2) has been recently obtained in [6] for square coefficients (and unknowns) all having the same size. Therefore, up to our knowledge, the only open question for (1) and (2) is to characterize the uniqueness of solution allowing for rectangular coefficients. The goal of this paper is to give a solution to this problem.

We emphasize that, for Equation (2), the approach followed in [6] for square coefficients cannot be applied in a straightforward manner in the rectangular case. Indeed, that approach is based on the characterization of uniqueness of solution of the $\star$-Sylvester equation $AX + X^\star D = 0$ provided in [3,11], which is valid only for $A, D, X \in \mathbb{C}^{n\times n}$. However, our approach will allow us to extend that characterization to the case of rectangular coefficients.

The most interesting feature of our characterization is the appearance of some peculiar additional constraints that do not have the same form as the spectral conditions appearing in [4] and [6] for the square case (just compare Theorems 1 and 4 with Theorems 2 and 5).

1.1 The main results

In this section we state the main results of this paper, namely Theorems 2 and 5. The rest of the paper is devoted to prove these results.

For the sake of completeness, we include here the characterization for the uniqueness of solution of (1) for the case of square coefficients.

Theorem 1. [4, Th. 1] *Let $A, C \in \mathbb{C}^{m\times m}$ and $B, D \in \mathbb{C}^{n\times n}$. Then the equation $AXB - CXD = E$ has a unique solution, for any right-hand side E, if and only if the matrix pencils $A - \lambda C$ and $D - \lambda B$ are regular and they have disjoint spectra.*

Theorem 2 presents a characterization for the uniqueness of solution of Equation (1) with coefficients of arbitrary size, which is an extension of Theorem 1. The proof is given in Section 2.

Theorem 2. *Let $A, C \in \mathbb{C}^{p\times m}$, and $B, D \in \mathbb{C}^{n\times q}$. The equation*

$$AXB - CXD = E$$

has a unique solution $X \in \mathbb{C}^{m\times n}$, for any right hand side E, if and only if one of the following situations holds:

(a) $p = m, q = n$, and the pencils $A - \lambda C$ and $D - \lambda B$ are regular and they have disjoint spectra;

(b) $p = 2m, n = 2q$, and the matrices $\begin{bmatrix} A & C \end{bmatrix}$ and $\begin{bmatrix} D & B \end{bmatrix}$ are invertible;

(c) $m = 2p, q = 2m$, and the matrices $\begin{bmatrix} A^\top & C^\top \end{bmatrix}$ and $\begin{bmatrix} D^\top & B^\top \end{bmatrix}$ are invertible.

It is interesting to note that the solution of Equation (1), allowing for rectangular coefficients, has been considered in [12], but no explicit characterization for the uniqueness of solution is given in this reference. Also, [10, Th. 3.2] presents a computation of the solution space of (1) with $B = I$, depending on the Kronecker canonical form of $A - \lambda C$, but restricted to the case where this canonical form does not contain right singular blocks.

To state the characterization of the uniqueness of solution of Equation (2) (in Theorem 5) we first need to introduce some notation and tools.

Given a regular matrix pencil $\mathcal{P}(\lambda)$, by $\Lambda(\mathcal{P})$ we denote the spectrum of $\mathcal{P}$, and $m_\lambda(\mathcal{P})$ denotes the algebraic multiplicity of the eigenvalue λ in $\mathcal{P}$. If M is a square matrix, by $\Lambda(M)$ and $m_\lambda(M)$ we denote, respectively, the spectrum of M and the algebraic multiplicity of λ as an eigenvalue of M.

We also recall the following notion, which plays a central role in Theorem 5.

Definition 3. (Reciprocal free and $*$-reciprocal free set) [3, 11]. Let $\mathcal{S}$ be a subset of $\mathbb{C} \cup \{\infty\}$. We say that $\mathcal{S}$ is

(a) reciprocal free if $\lambda \neq \mu^{-1}$, for all $\lambda, \mu \in \mathcal{S}$;

(b) $*$-reciprocal free if $\lambda \neq (\overline{\mu})^{-1}$, for all $\lambda, \mu \in \mathcal{S}$.

This definition includes the values $\lambda = 0, \infty$, with the customary assumption $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

Before stating the condition for the uniqueness of solution in the general case, we recall here the main result in [6], namely the characterization of the uniqueness of solution of (2) when all coefficients are square and with the same size.

Theorem 4. [6, Th. 15] *Let $A, B, C, D \in \mathbb{C}^{n\times n}$ and let $\mathcal{Q}(\lambda) = \left[\begin{smallmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{smallmatrix}\right]$. Then the equation $AXB + CX^\star D = E$ has a unique solution, for any right-hand side E, if and only if $\mathcal{Q}(\lambda)$ is regular and:*

- *If $\star = \top$, $\Lambda(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{Q})$ is $*$-reciprocal free.*

If we allow for rectangular coefficient matrices, then several subtleties arise, and in the end, they will result in additional restrictions in the pencil $\mathcal{Q}(\lambda)$. More precisely, in the case where $p = m$, that is, $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{m\times n}, D \in \mathbb{C}^{m\times n}$, the spectrum of the matrix pencil (3) contains the infinite eigenvalue, with multiplicity at least $|m - n|$. Then, in this case we denote by $\widehat{\Lambda}(\mathcal{Q})$ the following set obtained from $\Lambda(\mathcal{Q})$:

$$\widehat{\Lambda}(\mathcal{Q}) := \begin{cases} \Lambda(\mathcal{Q}), & \text{if } m_\infty(\mathcal{Q}) > |m-n|, \\ \Lambda(\mathcal{Q}) \setminus \{\infty\}, & \text{if } m_\infty(\mathcal{Q}) = |m-n|. \end{cases}$$

However, if $p = n$, that is $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times m}, D \in \mathbb{C}^{n\times n}$, the spectrum of the matrix pencil (3) contains the zero eigenvalue, with multiplicity at least $|m-n|$. Then, in this case we denote by $\widetilde{\Lambda}(\mathcal{Q})$ the following set obtained from $\Lambda(\mathcal{Q})$:

$$\widetilde{\Lambda}(\mathcal{Q}) := \begin{cases} \Lambda(\mathcal{Q}), & \text{if } m_0(\mathcal{Q}) > |m-n| \\ \Lambda(Q) \setminus \{0\}, & \text{if } m_0(\mathcal{Q}) = |m-n| \end{cases} .$$

The presence of these additional zero/infinity eigenvalues of $\mathcal{Q}(\lambda)$ in (3) is due to the "rectangularity" of either the diagonal blocks C, D or the anti-diagonal blocks A and B. Following [8], based on the theory developed in [13], these extra zero/infinity eigenvalues are called *dimension induced* eigenvalues. The sets $\widehat{\Lambda}(\mathcal{Q})$ and $\widetilde{\Lambda}(\mathcal{Q})$ are referred to as the set of *core eigenvalues*.

With these considerations in mind, we can state the other main result of this paper, which is an extension of Theorem 4 and which will be proved in Section 3.

Theorem 5. *Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{m\times q}$. Set $\mathcal{Q}(\lambda) := \left[\begin{smallmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{smallmatrix}\right]$. Then the equation*

$$AXB + CX^\star D = E$$

has a unique solution, for any right-hand side E, if and only if $\mathcal{Q}(\lambda)$ is regular and one of the following situations holds:

(1) *$p = m \neq n = q$, either $m > n$ and B is invertible or $m < n$ and A is invertible, and*

- *If $\star = \top$, $\widehat{\Lambda}(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\widehat{\Lambda}(\mathcal{Q})$ is $*$-reciprocal free.*

(2) *$p = n \neq m = q$, either $m > n$ and C is invertible or $m < n$ and D is invertible, and*

- *If $\star = \top$, $\widetilde{\Lambda}(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\widetilde{\Lambda}(\mathcal{Q})$ is $*$-reciprocal free.*

(3) *$p = m = n = q$, and*

- *If $\star = \top$, $\Lambda(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{Q})$ is $*$-reciprocal free.*

Throughout the paper we denote by I the identity matrix of appropriate size, and by $M^{-\star}$ we denote the inverse of the matrix $M^\star$, for an invertible matrix M.

The *reversal pencil* of the matrix pencil $\mathcal{P}(\lambda) = \lambda M + N$ is the pencil $\operatorname{rev}\mathcal{P}(\lambda) := \lambda N + M$. The pencil $\mathcal{P}(\lambda)$ has an infinite eigenvalue if and only if $\operatorname{rev}\mathcal{P}(\lambda)$ has the zero eigenvalue. The multiplicity (either algebraic or geometric) of the infinite eigenvalue in $\mathcal{P}(\lambda)$ is the multiplicity of the zero eigenvalue in $\operatorname{rev}\mathcal{P}(\lambda)$.

1.2 Vectorization

Both equations (1) and (2) can be considered as linear systems in the entries of the unknown matrix X. The natural approach to get such systems is applying the vectorization (vec) operator [9, §4.3]. Let $X \in \mathbb{C}^{m\times n}$ in both equations.

Regarding the generalized Sylvester equation (1), it must be $A, C \in \mathbb{C}^{p\times m}$, $B, D \in \mathbb{C}^{n\times q}$, and $E \in \mathbb{C}^{p\times q}$. Then, after applying the vec operator we get a linear system in the unknown $\operatorname{vec}(X)$ whose coefficient matrix $M = B^\top \otimes A - D^\top \otimes C$ has size $(pq) \times (mn)$.

For equation (2), we set $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}, E \in \mathbb{C}^{p\times q}$. In the case $\star = \top$, after applying the vec operator we obtain a linear equation $M \operatorname{vec}(X) = \operatorname{vec}(E)$, with $M \in \mathbb{C}^{(pq)\times(mn)}$ given by

$$M = B^\top \otimes A + (D^\top \otimes C)\Pi, \tag{4}$$

where Π is a permutation matrix associated to the transposition [9, Equation 4.3.9b].

In the case $\star = *$, more care is needed, since the system obtained by vectorization is not linear over $\mathbb{C}$, due to the presence of conjugations. Nevertheless, we can separate real and imaginary parts as in [6, §1.1] and write it as a $\mathbb{R}$-linear system of size $(2pq) \times (2mn)$ in $Y = \operatorname{vec}(\begin{bmatrix} \operatorname{re}(X) & \operatorname{im}(X) \end{bmatrix})$.

A different reformulation of the $\star = *$ case as a linear system is the following.

Lemma 6. *Equation* (2)*, with $\star = *$, has a unique solution if and only if the linear system of equations*

$$\begin{aligned} AXB + CYD &= 0, \\ D^*XC^* + B^*YA^* &= 0, \end{aligned} \tag{5}$$

has a unique solution.

Proof. Let us first assume that (2) has a nonzero solution X. Then this gives a nonzero solution (X, X^*) of (5).

To prove the converse, let (X, Y) be a nonzero solution of (5). Then, the matrix $X + Y^*$ is a solution of (2). If $X + Y^*$ is zero, then $Y = -X^*$, and in this case $\mathfrak{i}X$ is a nonzero solution of (2), with $\mathfrak{i} = \sqrt{-1}$. □

The matrix associated to (5) after applying the vec operator is

$$M = \begin{bmatrix} B^\top \otimes A & D^\top \otimes C \\ \overline{C} \otimes D^* & \overline{A} \otimes B^* \end{bmatrix}. \tag{6}$$

The fact that equations (1) and (2) are equivalent to linear systems has two important consequences. The first one is that (1) and (2) can have a unique solution, for any right-hand-side, only if the coefficient matrix of the linear system is square, that is, $mn = pq$. The second one is that the uniqueness of solution does not depend on the right-hand side: both (1) and (2) have a unique solution for any E if and only if the corresponding homogeneous equations

$$AXB - CXD = 0 \tag{7}$$

and

$$AXB + CX^{\star}D = 0 \tag{8}$$

have only the trivial solution $X = 0$. Hence, from now on, we focus on equations (7) and (8) instead of (1) and (2), respectively.

2 The generalized Sylvester equation

Here we provide the proof of Theorem 2, which gives a complete characterization of the uniqueness of solution of (1) for any right-hand side.

Proof of Theorem 2. As mentioned in Section 1.2, in order for (1) to have a unique solution for any right-hand side E, it must be $mn = pq$. Conversely, in all cases (a)–(c) in the statement of Theorem 2 we have $mn = pq$. Hence, we can restrict ourselves to this case and, as mentioned above, we may focus on the homogenous equation (7).

Let us first assume that $p > m$, which implies $q < n$. There exist two invertible matrices $Q \in \mathbb{C}^{n\times n}$ and $U \in \mathbb{C}^{p\times p}$ such that

$$QD = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \qquad UA = \begin{bmatrix} A_1 \\ 0 \end{bmatrix},$$

where $D_1 \in \mathbb{C}^{q\times q}$ and $A_1 \in \mathbb{C}^{m\times m}$. Pre-multiplying in (7) by U and setting $\widetilde{X} = XQ^{-1}$ we get the equivalent equation

$$\widetilde{A}\widetilde{X}\widetilde{B} - \widetilde{C}\widetilde{X}\widetilde{D} = 0, \tag{9}$$

where

$$\widetilde{A} = UA = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \quad \widetilde{B} = QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \widetilde{C} = UC = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \widetilde{D} = QD = \begin{bmatrix} D_1 \\ 0 \end{bmatrix},$$

where the block partitions of C and B are conformal with the ones of A and D, respectively. If we partition $\widetilde{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, with $X_1 \in \mathbb{C}^{m\times q}$, then we can decouple Equation (9) into the equivalent system

$$\left\{ \begin{array}{rcl} A_1X_2B_2 + A_1X_1B_1 - C_1X_1D_1 & = & 0, \\ C_2X_1D_1 & = & 0, \end{array} \right. \tag{10}$$

which has a unique solution if and only if the original equation (7) has a unique solution.

If A_1 is singular, then there is some $u \neq 0$ with $A_1u = 0$, and (10) has a nonzero solution, say $(0, uv^\top)$, where v is a nonzero vector. If B_2 has nontrivial left kernel, namely $v \neq 0$ with $v^\top B_2 = 0$, then (10) has a nonzero solution, say $(0, uv^\top)$, where u is a nonzero vector. Similarly, if D_1 is singular or C_2 has a nontrivial right kernel, then the second equation in (10) has infinitely many solutions. Assuming further that A_1 is nonsingular and B_2

has full row rank, then (10) has infinitely many solutions, obtained by solving $X_2 = -A_1^{-1}(A_1X_1B_1 + C_1X_1D_1)B_2^\dagger$, where $B_2^\dagger$ is a right inverse of B_2, for any solution X_1 of the second equation.

In summary, if the system has a unique solution then A_1 and D_1 must be invertible and B_2 must have full row rank, while C_2 must have full column rank.

We know that if $B_2 \in \mathbb{C}^{(n-q)\times q}$ has full row rank then $q \geqslant n - q$, that is $2q \geqslant n$, which implies $p \leqslant 2m$ (since $pq = mn$). On the other hand, we know that if $C_2 \in \mathbb{C}^{(p-m)\times m}$ has full column rank then $m \leqslant p - m$, which implies $p \geqslant 2m$.

Gathering all this information we obtain that, under the assumption $p > m$, a necessary condition for the system (10) to have a unique solution is that $p = 2m$ and, moreover, A_1 and D_1 are invertible, B_2 has full row rank and thus is invertible (since, for $p = 2m$, it is a square matrix), and C_2 has full column rank and thus is invertible as well.

This necessary condition can be stated as: $p = 2m$ and $\begin{bmatrix} A & C \end{bmatrix}$ and $\begin{bmatrix} D & B \end{bmatrix}$ are invertible. Indeed, from

$$U \begin{bmatrix} A & C \end{bmatrix} = \left[\begin{array}{c|c} A_1 & C_1 \\ \hline 0 & C_2 \end{array}\right], \qquad Q \begin{bmatrix} D & B \end{bmatrix} = \left[\begin{array}{c|c} D_1 & B_1 \\ \hline 0 & B_2 \end{array}\right],$$

we have that $\begin{bmatrix} D & B \end{bmatrix}$ is invertible if and only if D_1 and B_2 are invertible, and $\begin{bmatrix} A & C \end{bmatrix}$ is invertible if and only if A_1 and C_2 are invertible. This proves (b).

When $p > m$, the conditions $p = 2m$ and $\begin{bmatrix} A & C \end{bmatrix}$ and $\begin{bmatrix} D & B \end{bmatrix}$ invertible are also sufficient. Indeed, if $p = 2m$ and A_1, B_2, C_2 and D_1 are invertible, then it is clear that system (10) has a unique solution and thus the original equation (7) has a unique solution.

Let us consider the case $p < m$. We replace (7) by the transposed equation

$$B^\top X^\top A^\top - D^\top X^\top C^\top = 0,$$

whose uniqueness of solution is equivalent to the uniqueness of solution of (7) and fits the previous case replacing q by p, $(B^\top, A^\top, D^\top, C^\top)$ by (A, B, C, D), and $X^\top$ by X.

Thus, when $p < m$, the equation (1) has a unique solution for any right hand side if and only if $q = 2m$, which implies $m = 2p$ and the matrices $\begin{bmatrix} C^\top & A^\top \end{bmatrix}$ and $\begin{bmatrix} B^\top & D^\top \end{bmatrix}$ are invertible, which is the same as requiring that $\begin{bmatrix} A^\top & C^\top \end{bmatrix}$ and $\begin{bmatrix} D^\top & B^\top \end{bmatrix}$ are invertible.

The remaining case, namely case (a), is Theorem 1. □

Theorem 2 has an interesting corollary regarding matrices which are given as a sum of two tensor products.

Corollary 7. *Let $F, H \in \mathbb{C}^{q\times n}$ and $G, K \in \mathbb{C}^{p\times m}$. The matrix*

$$F \otimes G + H \otimes K$$

is invertible if and only if $pq = mn$ and

(a) $p = m$ (and $q = n$) and the pencils $F - \lambda H$ and $K - \lambda G$ are regular and they have disjoint spectra;

(b) $p = 2m$ (and $n = 2q$) and the matrices $\begin{bmatrix} F^\top & H^\top \end{bmatrix}$ and $\begin{bmatrix} G & K \end{bmatrix}$ are invertible;

(c) $m = 2p$ (and $q = 2n$) and the matrices $\begin{bmatrix} F & H \end{bmatrix}$ and $\begin{bmatrix} G^\top & K^\top \end{bmatrix}$ are invertible.

3 The generalized $\star$-Sylvester equation

Here we provide a proof of Theorem 5, which gives a complete characterization of the uniqueness of solution of (2) for any right-hand side. We consider separately the case in which every coefficient is non-square, namely $p \notin \{m, n\}$, which is treated in Section 3.1, and the case in which two coefficients are square and two are nonsquare, namely $p \in \{m, n\}$, which is treated in Section 3.2. The case where all coefficients are square is Theorem 4.

3.1 The case $p \notin \{m, n\}$

In this section we show that Theorem 5 holds if $p \notin \{m, n\}$. Note that, because of the restriction $mn = pq$, this also implies that $q \notin \{m, n\}$, so this situation covers all instances of Theorem 5 where none of the coefficient matrices are square.

Lemma 8. *Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$. If $mn = pq$ and $p \notin \{m, n\}$ then $AXB + CX^\star D = 0$ has a nonzero solution.*

Proof. We have four cases, depending on whether p is smaller or larger than m and n.

1. $p < \min(m, n)$. There are two nonzero vectors u, v such that $Au = 0$ and $Cv = 0$, because of the dimensions of these two matrices. Then $X = uv^\star$ is a nonzero solution of (8).

2. If $p > \max(m, n)$, the identity $mn = pq$ implies $q < \min(m, n)$. Then, there are two nonzero vectors u, v such that $v^\star B = 0$, $u^\star D = 0$, and $X = uv^\star$ is a nonzero solution of (8).

3. $m < p < n$. In this case, and because of the identity $mn = pq$, we have $m < q < n$ as well. Therefore, $m < \min(p, q)$. In particular, there exist nonzero vectors u, v such that $u^\top A = 0$, $v^\top D^\top = 0$.

   Now we consider the cases:

   (a) $\star = \top$: As argued in Section 1.2, Equation (8) is equivalent to the linear system $M \operatorname{vec} X = 0$, with the matrix $M \in \mathbb{C}^{(mn)\times(mn)}$ as in (4). Then, $(v^\top \otimes u^\top)M = 0$, so M is singular and (8) has a nonzero solution.

(b) $\star = *$: As a consequence of Lemma 6, Equation (8) has a nonzero solution if and only if the (square) matrix (6) is singular. It is easy to verify that $\begin{bmatrix} v^\top \otimes u^\top & u^* \otimes v^* \end{bmatrix} M = 0$, so M is indeed singular.

4. $n < p < m$. By setting $Y = X^\star$, Equation (8) is equivalent to $CYD + AY^\star B = 0$, so we use the result for the previous case on this equation. □

3.2 The case $p \in \{m, n\}$

We have seen in Lemma 8 that if all coefficient matrices in (2) are rectangular, then (8) cannot have a unique solution when $mn = pq$. In order for (2) to have a unique solution, for any right-hand side E, then (8) must have a unique solution. Therefore, it remains to consider the cases $p = m$ and $p = n$. For the reader's convenience, we include a complete proof of Theorem 5.

Proof of Theorem 5. As in the proof of Theorem 2, we first show that we can restrict ourselves to the case $mn = pq$. In order for (2) to have a unique solution, for any right-hand side E, the matrix associated with equation (2) must be square, and this implies $mn = pq$, as explained in Section 1.2. Conversely, this condition, together with the ones in each case (1)–(3) in the statement, imply $mn = pq$.

We begin by considering the case $p \notin \{m, n\}$. As a direct consequence of Lemma 8, the statement is true in this case. More precisely, the solution of (8) is non-unique, and $\mathcal{Q}(\lambda)$ is singular because it is non-square. To see this, note that $\mathcal{Q}(\lambda)$ has size $(p + q) \times (m + n)$. If $p + q = m + n$ this fact, together with the identity $mn = pq$, would imply $\{m, n\} = \{p, q\}$, since both m, n and p, q are the roots of the same quadratic polynomial, namely $x^2 - (m + n)x + mn$.

It remains to consider the cases where either $p = m$ or $p = n$, which imply $q = n$ and $q = m$, respectively, due to the constraint $mn = pq$. Let us assume that $p = m > n$, so that D has more rows than columns, and there is some $u \neq 0$ such that $u^\star D = 0$. If B is singular, then there is some $v \neq 0$ such that $v^\star B = 0$. Therefore $X = uv^\star$ is a nontrivial solution of (8).

Assume now that (8) has a unique solution. Then, B is guaranteed to be nonsingular, and $AXB + CX^\star D = 0$ has a unique solution if and only if $AX + CX^\star DB^{-1} = 0$ has a unique solution. Moreover, we can find an invertible matrix Q such that

$$QDB^{-1} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \tag{11}$$

with $D_1 \in \mathbb{C}^{n\times n}$. This allows us to rewrite (8), after multiplying on the right by B^{-1}, and setting $Y = Q^{-\star}X$, in the equivalent form

$$AQ^\star \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + C \begin{bmatrix} Y_1^\star & Y_2^\star \end{bmatrix} \begin{bmatrix} D_1 \\ 0 \end{bmatrix} = 0, \tag{12}$$

where Y_1 has size $n \times n$ and Y_2 has size $(m-n) \times n$. If we partition $AQ^\star$ conformally as

$$AQ^\star = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \tag{13}$$

with $\widetilde{A}_{11} \in \mathbb{C}^{n\times n}, \widetilde{A}_{22} \in \mathbb{C}^{(m-n)\times(m-n)}$, then the block $\left[\begin{smallmatrix} \widetilde{A}_{12} \\ \widetilde{A}_{22} \end{smallmatrix}\right]$ has full column rank. If that were not the case we could find $Y_2 \neq 0$ such that

$$\begin{bmatrix} \widetilde{A}_{12} \\ \widetilde{A}_{22} \end{bmatrix} Y_2 = 0,$$

and this would imply

$$AQ^\star \begin{bmatrix} 0 \\ Y_2 \end{bmatrix} + C \begin{bmatrix} 0 & Y_2^\star \end{bmatrix} \begin{bmatrix} D_1 \\ 0 \end{bmatrix} = 0,$$

so equation (12) would have a nontrivial solution. Then, there is an invertible matrix U such that

$$UAQ^\star = U \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} = \begin{bmatrix} \widehat{A}_{11} & 0 \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix},$$

with $\widehat{A}_{22} \in \mathbb{C}^{(m-n)\times(m-n)}$ nonsingular. If we set $UC = \left[\begin{smallmatrix} \widehat{C}_1 \\ \widehat{C}_2 \end{smallmatrix}\right]$, with $\widehat{C}_1 \in \mathbb{C}^{n\times n}, \widehat{C}_2 \in \mathbb{C}^{(m-n)\times n}$ then, after multiplying on the left by U, (12) is equivalent to the system

$$\begin{array}{rcl} \widehat{A}_{11}Y_1 + \widehat{C}_1 Y_1^\star D_1 & = & 0, \\ \widehat{A}_{22}Y_2 & = & -(\widehat{A}_{21}Y_1 + \widehat{C}_2 Y_1^\star D_1). \end{array}$$

Since $\widehat{A}_{22}$ is nonsingular, the above system has a unique solution if and only if the first equation

$$\widehat{A}_{11}Y_1 + \widehat{C}_1 Y_1^\star D_1 = 0 \tag{14}$$

has a unique solution.

We are now ready to relate the uniqueness of solution of (8) to the spectral properties of the pencil $\mathcal{Q}(\lambda)$ in the statement. We perform the following left and right invertible transformations to $\mathcal{Q}(\lambda)$:

$$\begin{bmatrix} B^{-\star} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \begin{bmatrix} Q^\star & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \lambda D_1^\star & 0 & I \\ \widehat{A}_{11} & 0 & \lambda\widehat{C}_1 \\ \widehat{A}_{21} & \widehat{A}_{22} & \lambda\widehat{C}_2 \end{bmatrix}. \tag{15}$$

Set

$$\widehat{\mathcal{Q}}(\lambda) = \begin{bmatrix} \lambda D_1^\star & I \\ \widehat{A}_{11} & \lambda\widehat{C}_1 \end{bmatrix}.$$

Then, by (15),

$$\det \mathcal{Q}(\lambda) = \alpha \det \widehat{\mathcal{Q}}(\lambda).$$

where $\alpha = (-1)^{m-n}(\det \widehat{A}_{22})/(\det B^{-\star} \det U \det Q^{\star})$ is a nonzero constant.

Since all coefficient matrices in (14) are square and with the same size, namely $n \times n$, Theorem 4 implies that $\widehat{\mathcal{Q}}(\lambda)$ is regular (so $\mathcal{Q}(\lambda)$ is regular as well) and

- If $\star = \top$, $\Lambda(\widehat{\mathcal{Q}}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\widehat{\mathcal{Q}}) = m_{-1}(\widehat{\mathcal{Q}}) \leqslant 1$.
- If $\star = *$, $\Lambda(\widehat{\mathcal{Q}})$ is $*$-reciprocal free.

Note that (15) implies that $\widehat{\Lambda}(\mathcal{Q}) = \Lambda(\widehat{\mathcal{Q}})$, so the previous two conditions are equivalent to the conditions on the spectrum of $\mathcal{Q}(\lambda)$ in the statement.

To prove the converse, let us assume that B is invertible, and that $\mathcal{Q}(\lambda)$ is regular and its spectrum satisfies the conditions in the statement. Then we can define the matrix Q as in (11) and we arrive to (13). Again, the block $\left[\begin{smallmatrix} \widetilde{A}_{12} \\ \widetilde{A}_{22} \end{smallmatrix}\right]$ has full column rank since, otherwise, the pencil $\mathcal{Q}(\lambda)$ would be singular. This is an immediate consequence of the identity:

$$\begin{bmatrix} B^{-\star} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda D^{\star} & B^{\star} \\ A & \lambda C \end{bmatrix} \begin{bmatrix} Q^{\star} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \lambda D_1^{\star} & 0 & I \\ \widetilde{A}_{11} & \widetilde{A}_{12} & \lambda C_1 \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \lambda C_2 \end{bmatrix}.$$

Proceeding as before, we conclude, that (8) is equivalent to (14). By Theorem 4, the hypotheses on $\mathcal{Q}(\lambda)$, and the fact that $\widehat{\Lambda}(\mathcal{Q}) = \Lambda(\widehat{\mathcal{Q}})$, we conclude that (14) has a unique solution, and this implies that (8) has a unique solution.

Now, let us assume that $m < n$. After applying the $\star$ operator in (8) and setting $Y = X^{\star}$, we arrive to the equivalent equation $B^{\star} Y A^{\star} + D^{\star} Y^{\star} C^{\star} = 0$. This equation is of the form (8), with the coefficients of the first summand being square, $B^{\star} \in \mathbb{C}^{n \times n}, A^{\star} \in \mathbb{C}^{m \times m}$ and $n > m$, so we are in the same conditions as before, and A needs to be invertible. Applying the result just proved for this case, we get that the pencil

$$\widetilde{\mathcal{Q}}(\lambda) = \begin{bmatrix} \lambda C & A \\ B^{\star} & \lambda D^{\star} \end{bmatrix}$$

satisfies the conditions in the statement. But, since

$$\widetilde{\mathcal{Q}}(\lambda) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{Q}(\lambda) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

then $\mathcal{Q}(\lambda)$ satisfies these conditions as well.

For the case $p = n \neq m$, we apply the $\star$ operator in (8), and we arrive to the equivalent equation $D^{\star} X C^{\star} + B^{\star} X^{\star} A^{\star} = 0$, whose coefficients are in the conditions of case 1. The pencil associated to this last equation is

$$\widehat{\mathcal{Q}}(\lambda) = \begin{bmatrix} \lambda A & C \\ D^{\star} & \lambda B^{\star} \end{bmatrix}.$$

This pencil is the reversal of the pencil:

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{Q}(\lambda),$$

so $\Lambda(\widehat{\mathcal{Q}}) = \Lambda^{-1}(\mathcal{Q}) := \{\lambda^{-1} : \lambda \in \Lambda(\mathcal{Q})\}$, including multiplicities. In particular, the conditions on being $(*)$-reciprocal free in the statement are the same for both pencils, and the roles of the zero and the infinite eigenvalue are exchanged.

Finally, the case $p = m = n$ is Theorem 4, proved in [6]. □

Remark 9. The conditions $n = q$ in part 1, and $m = q$ in part 2 in Theorem 5 are redundant, but we have included them for emphasis. These conditions are a consequence of the fact that $\mathcal{Q}(\lambda)$ in (3) is regular and the other conditions on the size, namely $p = m$ and $p = n$, respectively. As indicated in the proof of Theorem 5, since $\mathcal{Q}(\lambda)$ has size $(p+q) \times (m+n)$, if it is regular, it must be, in particular, square, and this implies $m + n = p + q$.

3.3 Necessity of the invertibility conditions

The characterization for the uniqueness of solution of (2) in Theorem 5 involves, in cases 1 and 2, the invertibility of some of the coefficient matrices. One might wonder if these conditions are really needed, or whether they could be stated in terms of spectral properties of the pencil $\mathcal{Q}(\lambda)$. However, the following example shows that the uniqueness of solution does not depend solely on the eigenvalues of $\mathcal{Q}(\lambda)$. Consider the following generalized $\top$-Sylvester equations (the same example works for the $\star = *$ case):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0, \tag{16}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0. \tag{17}$$

The above equations have associated pencils defined as follows:

$$\mathcal{Q}_1(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 0 \\ \hline 1 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right], \qquad \mathcal{Q}_2(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 1 \\ \hline 0 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right].$$

The above pencils are the same up to row and column permutations, so they have the same eigenvalues. However, the corresponding generalized Sylvester equations (16)–(17) can be rewritten, respectively, as

$$x = 0, \qquad \text{and} \qquad x = y = 0.$$

Then (16) has infinitely many solutions, while (17) has a unique solution.

3.4 Some corollaries

The characterization given in Theorem 5 depends on spectral properties of the pencil $\mathcal{Q}(\lambda)$ in (3), which has twice the size of the coefficient matrices of Equation (2). With some additional effort, we can provide a characterization in terms of pencils with exactly the same size.

Corollary 10. *Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{m\times q}$. Then the equation $AXB + CX^\star D = E$ has a unique solution, for any right-hand side E, if and only if one of the following situations holds:*

(1) *$p = m \leqslant n = q$, A is invertible, the pencil $\mathcal{P}_1(\lambda) := B^\star - \lambda D^\star A^{-1} C$ is regular and*

- *If $\star = \top$, $\Lambda(\mathcal{P}_1) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_1) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{P}_1)$ is $*$-reciprocal free.*

(2) *$p = m \geqslant n = q$, B is invertible, the pencil $\mathcal{P}_2(\lambda) := A^\star - \lambda D B^{-1} C^\star$ is regular and*

- *If $\star = \top$, $\Lambda(\mathcal{P}_2) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_2) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{P}_2)$ is $*$-reciprocal free.*

(3) *$p = n \leqslant m = q$, C is invertible, the pencil $\mathcal{P}_3(\lambda) := D^\star - \lambda B^\star C^{-1} A$ is regular and*

- *If $\star = \top$, $\Lambda(\mathcal{P}_3) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_3) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{P}_3)$ is $*$-reciprocal free.*

(4) *$p = n \geqslant m = q$, D is invertible, the pencil $\mathcal{P}_4(\lambda) := C^\star - \lambda B D^{-1} A^\star$ is regular and*

- *If $\star = \top$, $\Lambda(\mathcal{P}_4) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_4) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{P}_4)$ is $*$-reciprocal free.*

Proof. Let us assume first that (2) has a unique solution, for any right-hand side E. Then Theorem 5 implies that at least one of the following situations holds: (C1) $p = m < n = q$ and A is invertible, (C2) $p = m > n = q$ and B is invertible, (C3) $p = n < m = q$ and D is invertible, (C4) $p = n > m = q$ and C is invertible, or (C5) $p = m = n = q$ and at least one of A, B, C, D is invertible. Let us first assume that case (C1) holds. We can perform the following unimodular equivalence on $\mathcal{Q}(\lambda)$ (which preserves the finite spectrum):

$$\begin{bmatrix} I & -\lambda D^\star A^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} = \begin{bmatrix} 0 & B^\star - \lambda^2 D^\star A^{-1} C \\ A & \lambda C \end{bmatrix}. \tag{18}$$

This shows, in particular, that $\mathcal{Q}(\lambda)$ is regular if and only if $\mathcal{P}_1(\lambda)$ is regular. Moreover, we can also perform the following strict equivalence transformation to rev $\mathcal{Q}(\lambda)$:

$$\begin{bmatrix} D^\star & \lambda B^\star \\ \lambda A & C \end{bmatrix} \begin{bmatrix} A^{-1}C & A^{-1} \\ -\lambda I & 0 \end{bmatrix} = \begin{bmatrix} D^\star A^{-1}C - \lambda^2 B^\star & D^\star A^{-1} \\ 0 & \lambda I \end{bmatrix}. \tag{19}$$

Taking determinants in (19) we arrive to

$$\det(\operatorname{rev} \mathcal{Q}(\lambda)) = \pm\lambda^{m-n} \det(A^{-1}) \det(\operatorname{rev} \mathcal{P}_1(\lambda)). \tag{20}$$

Equations (18) and (20) show that $\widehat{\Lambda}(\mathcal{Q}) = \sqrt{\Lambda(\mathcal{P}_1)} := \left\{\mu : \mu^2 \in \Lambda(\mathcal{P}_1)\right\}$, including multiplicities. Then Theorem 5 implies that claim 1 in the statement holds.

If case (C2) holds, then we apply the $\star$ operator in (2) and apply the previous arguments to the new equation and its corresponding pencil.

If case (C3) holds, then after introducing the change of variables $Y = X^\star$, the roles of A, B and C, D are exchanged, so we apply the same arguments as in case (C1) to the corresponding pencil, $\mathcal{P}_3(\lambda)$.

In case (C4), we apply the $\star$ operator in (2) and introduce the change of variables $Y = X^\star$. Then we apply the same arguments as for case (C1) to the pencil corresponding to this new equation.

Finally, if we are in case (C5), at least one of 1–4 in the statement holds, and we are done.

To prove the converse, let us assume that any of 1–4 in the statement holds. Then, reversing the previous arguments, we can conclude that at least one of the situations 1–3 in the statement of Theorem 5 occurs, and Theorem 5 implies that (2) has a unique solution, for any right-hand side. □

As another corollary of Theorem 5 we get an extension of [3, Lemma 5.10] and [11, Lemma 8] for the $\star$-Sylvester equation $AX + X^\star D = E$ (see also [5, Th. 10, Th. 11]) to the case of rectangular coefficients.

Corollary 11. *Let $A \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{n\times m}$. Then the equation $AX + X^\star D = E$ has a unique solution, for any right-hand side E, if and only if the matrix pencil $\mathcal{P}(\lambda) = A - \lambda D^\star$ is regular and:*

- *If $\star = \top$, $\Lambda(\mathcal{P}) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{P})$ is $*$-reciprocal free.*

For the Sylvester equation $AX - XD = E$, in order to have a unique solution for any right hand side, non-square coefficients are not allowed, since otherwise the equation would not be well-defined.

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