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A DEFLATION APPROACH FOR LARGE-SCALE LUR'E EQUATIONS*

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Abstract. We present an approach to the determination of the stabilizing solution of Lur'e matrix equations. We show that the knowledge of a certain deflating subspace of an even matrix pencil may lead to Lur'e equations which are defined on some subspace, the so-called "projected Lur'e equations." These projected Lur'e equations are shown to be equivalent to projected Riccati equations, if the deflating subspace contains the subspace corresponding to infinite eigenvalues. This result leads to a novel numerical algorithm that basically consists of two steps. First we determine the deflating subspace corresponding to infinite eigenvalues using an algorithm based on the so-called "neutral Wong sequences," which requires a moderate number of kernel computations; then we solve the resulting projected Riccati equations. Altogether this method can deliver solutions in low-rank factored form, it is applicable for large-scale Lur'e equations and exploits possible sparsity of the matrix coefficients.

Key words. Lur'e equations, Riccati equations, deflating subspaces, even matrix pencils, Newton–Kleinman method, ADI iteration

AMS subject classifications. 15A24, 15A22, 15A18, 49N10, 93A15, 93B60

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1. Introduction. For given matrices $A \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, and Hermitian $Q \in \mathbb{C}^{n,n}$ $R, J \in \mathbb{C}^{m,m}$, where J is a *signature matrix* (i.e., $J = \operatorname{diag}(\pm 1, \dots, \pm 1)$), we consider *Lur'e equations*

$$\begin{aligned} A^*X + XA + Q &= K^*JK, \\ XB + S &= K^*JL, \\ R &= L^*JL, \end{aligned} \tag{1.1}$$

which have to be solved for the triple $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{m,n} \times \mathbb{C}^{m,m}$ with Hermitian X and $p = \operatorname{rank}[K\ ,\ L]$ as small as possible. For sake of simplicity, we will call X a solution of the Lur'e equations, if there exist K and L such that (1.1) holds true.

These types of equations arise in *J-spectral factorization*. That is, for a given $m \times m$-valued rational rational function $\Phi(s) \in \mathbb{C}(s)^{m,m}$ which is *para-Hermitian* (that is, Φ is Hermitian on the imaginary axis), one seeks for rational function $\Psi(s) \in \mathbb{C}(s)^{m,m}$ with

$$\Psi(\overline{s})^* J \Psi(s) = \Phi(s). \tag{1.2}$$

More precisely, for

$$\Phi(s) = R + B^*(-sI - A^*)^{-1}Q(sI - A)^{-1}B + B^*(-sI - A^*)^{-1}S + S^*(sI - A)^{-1}B, \tag{1.3}$$

a simple calculation yields that solutions of the Lur'e equations give rise to a factorization (1.2) with

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$$\Psi(s) = L + K(sI - A)^{-1}B. \tag{1.4}$$

The rational function $\Psi(s)$ is called a *J-spectral factor*; $\Phi(s)$ is often referred to as the *spectral density function*, or *Popov function*.

The problem of J-spectral factorization plays a key role in $\mathcal{H}_\infty$-controller design [23] and also occurs in the frequency domain consideration of differential games [5]. An important special case is J being the identity: Here, the spectral factorization problem is a crucial tool for linear-quadratic optimal control. Namely, the spectral factorization problem can be seen as the frequency domain counterpart of the minimization (resp., "infimization") of the cost functional

$$\mathcal{J}(u(\cdot), x_0) = \frac{1}{2}\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{1.5}$$

subject to the constraint defined by the ordinary differential equation $\dot{x}(t) = Ax(t) + Bu(t)$ with initial and end conditions $x(0) = x_0$, $\lim_{t\to\infty} x(t) = 0$ [50]. Indeed, it can be shown that the optimal control and the optimal value of the cost functional can be determined by means of a certain solution of the Lur'e equations.

In the case where the input is "fully weighted," i.e., the matrix R is invertible, then the unknown matrices K and L can be eliminated and one obtains an *algebraic Riccati equation (ARE)*

$$A^*X + XA - (XB + S)R^{-1}(XB + S)^* + Q = 0. \tag{1.6}$$

While, e.g., in linear-quadratic optimal control, the invertibility of R is often a reasonable assumption, there exist various other important applications for Lur'e equations with possibly singular R: In $\mathcal{H}_\infty$ control, invertibility of R corresponds to full-rank properties of certain feedthrough terms of the plant [21, 19, 20]. This cannot always be justified by practice. Furthermore, singular problems also occur in balancing-related model order reduction: The methods of *positive real balanced truncation* and *bounded real balanced truncation* [13, 24, 34, 36, 39] require the numerical solution of large-scale Lur'e equations. Here the singularity of R is often a structural property of the system to be analyzed [38] and can therefore not be excluded by arguments of genericity.

Though the numerical solution of (especially large-scale) algebraic Riccati equations is still the subject of present research, this field can be considered as widely well understood [6, 12]. For the case of definite R, the *Newton–Kleinman method* [31] is a popular choice mainly because of the following reasons: a starting value for the iteration can be easily determined by solving a simple stabilization problem; the method is usually quadratically convergent; and, last but not least, it can be reformulated such that the iterates X_i appear in *low rank factored form* $X^{(i)} = Z^{(i)}(Z^{(i)})^*$ for some $Z^{(i)} \in \mathbb{C}^{n,k_i}$ with $k_i \ll n$ [7]. The latter property enables a significantly less memory-consuming implementation, and, furthermore, factorizations of the solutions are required anyway in many applications, such as balancing-related model order reduction [8, 24]. If R is invertible but indefinite, the numerical solution of the Riccati equation (1.6) becomes more involved. In [14], the Newton–Kleinman iteration is considered. However, the starting value has to fulfill an additional inequality constraint and is, in general, difficult to compute. A completely different kind of iteration has been proposed in [33], where a recursion is presented that requires the solution of a Riccati equation with definite quadratic term. Quadratic convergence is proven.

While the numerical analysis for algebraic Riccati equations has achieved a considerably advanced level [6, 12, 41], the case of singular R has been treated stepmotherly. The existing results are either purely analytical [16, 17, 40], or describe numerical methods that are feasible only for small dense problems. Before our approach is presented, let us briefly review some known theoretical and numerical approaches to the spectral factorization and optimal control problem:

(a) The paper [27] considers a completely different type of equation to approach the linear-quadratic optimal control problem. It is seeked for a quintuple $(r, X, V, F, S) \in \mathbb{N} \times \mathbb{C}^{n,n} \times \mathbb{C}^{n,r} \times \mathbb{C}^{m,n}$, such that V^*XV is Hermitian, V has full column rank, the spectrum of S is located in the complex left half plane, and

$$\begin{aligned} &\begin{bmatrix} A^*X + XA + Q & XB + S \\ S^* + B^*X & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} V = 0, \\ &(A + BF)V = VS. \end{aligned}$$

It is shown that the space $\operatorname{im} V$ determines the set of initial states $x_0 \in \mathbb{R}^n$ for which the optimal control problem (1.5) has a solution. It is only possible in special cases to relate these solutions to the spectral factorization problem.

(b) In [26, 48, 28], the Lur'e equations (1.1) with $J = I_m$ are transformed to a *constrained Riccati equation*

$$\begin{aligned} &A^*X + XA - (XB + S)R^+(XB + S)^* + Q = 0, \\ &\ker R \subset \ker(XB + S). \end{aligned} \tag{1.7}$$

Solvability criteria are presented in terms of spectral properties of so-called *extended Hamiltonian matrix pencils*. In certain special cases, solutions of the constrained Riccati equation give rise to solutions of the spectral factorization problem. Later on we will present some further comments on the relation to the equations considered in this work.

(c) The most common approach to the numerical solution of Lur'e equations with $J = I_m$ in engineering practice is *regularization*, i.e., the slight perturbation of R by εI_m such that $R + \varepsilon I$ is invertible. The corresponding perturbed Lur'e equations are now equivalent to the Riccati equation

$$A^*X_\varepsilon + X_\varepsilon A - (XB + S)(R + \varepsilon I)^{-1}(X_\varepsilon B + S)^* + Q = 0. \tag{1.8}$$

It is shown in [30, 44] that convergence of desired solutions X_ε then converge as ε tends to zero.

(d) The works [29, 47] present an successive technique for the elimination of variables corresponding to $\ker R$. By performing an orthogonal transformation of R, and an accordant transformation of L, the equations can be divided into a "regular part" and a "singular part." The latter leads to an explicit equation for a part of the matrix K. Plugging this part into (1.1), one obtains Lur'e equations of slightly smaller size. After a finite number of steps this leads to an algebraic Riccati equation. This also gives an equivalent solvability criterion that is obtained by the feasibility of this iteration. The regularization approach has two essential disadvantages: so far, no estimates for the perturbation error $\|X - X_\varepsilon\|$ have been found, and even convergence rates are unknown. Furthermore, the numerical sensitivity of the Riccati equation (1.8) increases drastically as ε tends to 0.

(e) Recently, the *structure-preserving doubling algorithm (SDA)* [15] was extended to a certain class of Lur'e equations [37]. Roughly speaking, the problem is transformed via Cayley transformation to the discrete-time case, and a power iteration leads to the desired solution. It is shown that this iteration converges linearly. Again, this method is only applicable to small-scale dense problems, and an additional restriction is that the associated pencil must be regular.

The approach presented in this work is related to 1) in the sense that the "singular part" of the Lur'e equation is extracted and, afterwards, an "inherent algebraic Riccati equation" is set up and solved. We make use of the results in [40, 3, 4], where it is shown that there exists a one-to-one correspondence between the solutions of Lur'e equations and certain deflating subspaces of the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}. \tag{1.9}$$

Based on these results, we show that the determination of deflating subspaces of the pencil (1.9) leads to the knowledge of the action of X on some subspace, that is,

$$X\check{V}_x = \check{V}_\mu \tag{1.10}$$

for some matrices $\check{V}_\mu, \check{V}_x \in \mathbb{C}^{n,\check{n}}$, which are constructed from a basis matrix of the deflating subspace of $s\mathcal{E} - \mathcal{A}$. Furthermore, we show that using the partial information in (1.10) we can reduce (1.1) to a system of *projected Lur'e equations*

$$\begin{aligned} \widetilde{A}^*\widetilde{X} + \widetilde{X}\widetilde{A} + \widetilde{Q} &= \widetilde{K}^*J\widetilde{K}, \qquad \widetilde{X} = \widetilde{X}^* = \Pi^*\widetilde{X}\Pi \in \mathbb{C}^{n,n}, \\ \widetilde{X}\widetilde{B} + \widetilde{S} &= \widetilde{K}^*J\widetilde{L}, \\ \widetilde{R} &= \widetilde{L}^*J\widetilde{L}, \end{aligned} \tag{1.11}$$

where $\Pi \in \mathbb{C}^{n,n}$ is a projector matrix (i.e., $\Pi^2 = \Pi$), the coefficients satisfy

$$\begin{gathered} \widetilde{A} = \Pi\widetilde{A}\Pi \in \mathbb{C}^{n,n}, \quad \widetilde{B} = \Pi\widetilde{B} \in \mathbb{C}^{n,\widetilde{m}}, \quad \widetilde{S} = \Pi^*\widetilde{S} \in \mathbb{C}^{n,\widetilde{m}}, \\ \widetilde{Q} = \widetilde{Q}^* = \Pi^*\widetilde{Q}\Pi \in \mathbb{C}^{n,n}, \quad \widetilde{R} = \widetilde{R}^* \in \mathbb{C}^{\widetilde{m},\widetilde{m}}. \end{gathered} \tag{1.12}$$

We prove that these projected Lur'e equations are implicitly equivalent to a Riccati equation as long as our deflating subspace contains a certain part of the deflating subspace corresponding to the infinite eigenvalues. This implicit algebraic Riccati equation can be solved by slight reformulations of the known approaches for conventional algebraic Riccati equations.

In [46] the deflating subspace approach has been considered for a special Riccati equations (R positive definite, Q positive semidefinite, $S = 0$): The full ($m + n$-dimensional) deflating subspace determining the desired solution has been computed by transforming the associated pencil to *staircase form* of the pencil. This form can be achieved by multiplication of the pencil $s\mathcal{E} - \mathcal{A}$ from the left and from the right with unitary matrices. Note that, by using the extension of the staircase form to general (possibly singular) matrix pencils, the deflation approach via staircase form could, in principle, be generalized to general Lur'e equations. However, since the unitary matrices involved in the staircase algorithm are dense, this approach is not well suited to large-scale problems in which preserving the sparsity structure of A is crucial. By the approach presented here of deflating only a "small and critical part" we will be able to enable the numerical advantages of iterative methods also in the case of general Lur'e equations.

This paper is organized as follows. In the forthcoming section, we arrange the basic notation and present the fundamental facts about matrix pencils and their normal forms. In particular, we present fundamentals of deflating subspaces, give a constructive approach via so-called *Wong sequences*, and develop some extensions which are useful in later parts. Thereafter, in section 3, we briefly repeat some results about solution theory for Lur'e equations. In particular, the connection between solutions and deflating subspaces of the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.9) is highlighted. As well, we slightly extend this theory to projected Lur'e equations. In section 4 we develop the main theoretical preliminaries for the numerical method introduced in this work: Based on the concept of *partial solution* we present some results on the structure of the corresponding projected Lur'e equations. In particular, we give equivalent criteria on the deflated subspace for the possibility to reformulate the projected Lur'e equations (1.11) as projected Riccati equations. This theory enables us to formulate in section 5 a numerical algorithm for solution of Lur'e equations which first consists of determining a "critical deflating subspace of $s\mathcal{E} - \mathcal{A}$," and then an iterative solution of the obtained projected algebraic Riccati equation. This paper ends with section 6, where the presented numerical approach is tested by means of several numerical examples.

2. Matrix theoretic preliminaries.

2.1. Nomenclature. We adopt the following notation.

$\mathbb{N}$, $\mathbb{N}_0$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\lfloor x \rfloor$	=	$\max\{n \in \mathbb{N} \mid n \leq x\} \cup \{0\}$, integer part of $x \in \mathbb{R}$
$\mathbb{C}^+$, $\mathbb{C}^-$, $i\mathbb{R}$		set of complex numbers with negative (positive, zero) real part, respectively
$\overline{z}$, A^*, A^T		conjugate of $z \in \mathbb{C}$, Hermitian and transpose of $A \in \mathbb{C}^{m,n}$, respectively
$\mathrm{Gl}_n(\mathbb{C})$		the group of invertible real $n \times n$ matrices
$\mathbb{C}[s]$		the ring of polynomials with coefficients in $\mathbb{C}$
$\mathbb{C}(s)$		the quotient field of $\mathbb{C}[s]$
$R^{n,m}$		the set of $n \times m$ matrices with entries in a ring R
$\Vert M \Vert_F$		Frobenius norm of $M \in \mathbb{C}^{n,m}$
M^+		Moore–Penrose inverse of $M \in \mathbb{C}^{n,m}$ [25, Chapter 5.7]
$\underset{(>,\leq,<)}{M \geq N}$		$M - N$ is Hermitian and positive semidefinite (positive definite, negative semidefinite, negative definite).
$M^{-1}\mathcal{Y}$	=	$\{x \in \mathbb{C}^m \mid Mx \in \mathcal{Y}\}$, the preimage of the set $\mathcal{Y} \subseteq \mathbb{C}^n$ under $M \in \mathbb{C}^{n,m}$
$\mathcal{Y}^{M\perp}$		$\{x \in \mathbb{R}^n \mid x^*My = 0 \; \forall y \in \mathcal{Y}\}$, the M-orthogonal complement of the subspace $\mathcal{Y} \subseteq \mathbb{R}^n$ (where $M \in \mathbb{R}^{n,m}$)
$\mathrm{diag}(A_1, \ldots, A_k)$		block-diagonal matrix with entries $A_i \in \mathbb{C}^{m_i,n_i}$ for $m_i, n_i \in \mathbb{N}_0$, $i = 1, \ldots, k$ ($A \in \mathbb{C}^{m,n}$ for $n = n_1 + \cdots + n_k$, $m = m_1 + \cdots + m_k$)
$n_+(M)$, $n_-(M)$, $n_0(M)$		the number of positive, negative, and zero eigenvalues of the Hermitian matrix $M \in \mathbb{C}^{n,n}$

Moreover, an identity matrix of size $n \times n$ is denoted by I_n or simply by I, the zero $n \times m$ matrix is by $0_{n,m}$ or simply by 0. The symbol $e_i^{(n)}$ (or simply e_i) stands for the ith column of I_n. A matrix V is called a *basis matrix* for a subspace $\mathcal{V}$ if it has full column rank and $\operatorname{im} V = \mathcal{V}$. A matrix $J \in \mathbb{R}^{m,m}$ is called a *signature matrix* if it is diagonal with $J^2 = I_m$ (i.e., $J = \operatorname{diag}(\pm 1, \ldots, \pm 1)$).

We further introduce the special matrices $J_k, M_k, N_k \in \mathbb{R}^{k,k}$, $K_k, L_k \in \mathbb{R}^{k-1,k}$ for $k \in \mathbb{N}$, which are given by

$$J_k = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \quad K_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad L_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},$$
$$M_k = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{bmatrix}, \quad N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}. \tag{2.1}$$

2.2. Matrix pencils. Here we introduce some fundamentals of matrix pencils, i.e., first order matrix polynomials $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ with $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{M,N}$.

DEFINITION 2.1. *A matrix pencil $P(s) = s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ is called*
(i) regular *if $M = N$ and* $\operatorname{rank}_{\mathbb{C}(s)} P(s) = N$, *and*
(ii) even *if $P(\overline{s})^* = P(s)$, i.e., $\mathcal{E} = -\mathcal{E}^T$ and $\mathcal{A} = \mathcal{A}^T$.*

Many properties of a matrix pencil can be characterized in terms of the *Kronecker canonical form (KCF)*.

THEOREM 2.2 (see [22]). *For a matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$, there exist matrices $U_l \in \mathrm{Gl}_M(\mathbb{C})$, $U_r \in \mathrm{Gl}_N(\mathbb{C})$, such that*

$$U_l(s\mathcal{E} - \mathcal{A})U_r = \operatorname{diag}(\mathcal{C}_1(s), \ldots, \mathcal{C}_k(s)), \tag{2.2}$$

where each of the pencils $\mathcal{C}_j(s)$ is of one of the types presented in Table 2.1. *The numbers λ appearing in the blocks of type $K1$ are called the* (generalized) eigenvalues *of $sE - A$. Blocks of type $K2$ are said to be corresponding to* infinite eigenvalues.

A special modification of the KCF for even matrix pencils, the so-called *even Kronecker canonical form (EKCF)* is presented in [42]. Note that there is also an extension of this form such that realness is preserved [43].

THEOREM 2.3 (see [42]). *For an even matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$, there exists some $U \in \mathrm{Gl}_N(\mathbb{C})$ such that*

$$U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1(s), \ldots, \mathcal{D}_k(s)), \tag{2.3}$$

where each of the pencils $\mathcal{D}_j(s)$ is of one of the types presented in Table 2.2. *The numbers ε_j in the blocks of type $E2$ and $E3$ are called the* block signatures.

The KCF can be easily obtained from an EKCF by permuting rows and columns: The blocks of type E1 contains pairs $(\lambda, -\overline{\lambda})$ of generalized eigenvalues. In the case

TABLE 2.1
Block types in Kronecker canonical form (*with matrices as defined in* (2.1)).

Type	Size	$\mathcal{C}_j(s)$	Parameters
K1	$k_j \times k_j$	$(s-\lambda)I_{k_j} - N_{k_j}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}$
K2	$k_j \times k_j$	$sN_{k_j} - I_{k_j}$	$k_j \in \mathbb{N}$
K3	$(k_j - 1) \times k_j$	$sK_{k_j} - L_{k_j}$	$k_j \in \mathbb{N}$
K4	$k_j \times (k_j - 1)$	$sK_{k_j}^T - L_{k_j}^T$	$k_j \in \mathbb{N}$

TABLE 2.2
Block types in even Kronecker canonical form (with matrices as defined in (2.1)).

Type	Size	$\mathcal{D}_j(s)$	Parameters
E1	$2k_j \times 2k_j$	$\begin{bmatrix} 0_{k_j,k_j} & (\lambda-s)I_{k_j}-N_{k_j} \\ (\overline{\lambda}+s)I_{k_j}-N_{k_j}^T & 0_{k_j,k_j} \end{bmatrix}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}^+$
E2	$k_j \times k_j$	$\epsilon_j((-is-\omega)J_{k_j}+M_{k_j})$	$k_j \in \mathbb{N}, \omega \in \mathbb{R}$, $\epsilon_j \in \{-1,1\}$
E3	$k_j \times k_j$	$\epsilon_j(isM_{k_j}+J_{k_j})$	$k_j \in \mathbb{N}$, $\epsilon_j \in \{-1,1\}$
E4	$(2k_j-1)\times$ $(2k_j-1)$	$\begin{bmatrix} 0_{k_j,k_j} & isK_{k_j}^T + L_{k_j}^T \\ isK_{k_j}+L_{k_j} & 0_{k_j-1,k_j-1} \end{bmatrix}$	$k_j \in \mathbb{N}$

where $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$, nonimaginary eigenvalues even occur in quadruples $(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda})$. The blocks of types E2 and E3, respectively, correspond to the purely imaginary and infinite eigenvalues. Blocks of type E4 consist of a combination of blocks that are equivalent to those of types K3 and K4. Note that regularity of the pencil $s\mathcal{E}-\mathcal{A}$ is equivalent to the absence of blocks of type E4.

The following concept generalizes the notion of invariant subspaces to matrix pencils.

DEFINITION 2.4. *A subspace $\mathcal{V} \subset \mathbb{C}^N$ is called a* (right) deflating subspace *for the pencil $s\mathcal{E}-\mathcal{A} \in \mathbb{C}[s]^{M,N}$ if, for a basis matrix $V \in \mathbb{C}^{N,k}$ of $\mathcal{V}$, there exists some $l \in \mathbb{N}_0$, a matrix $W \in \mathbb{C}^{M,l}$, and a pencil $s\widetilde{E}-\widetilde{A} \in \mathbb{C}[s]^{l,k}$ with $\operatorname{rank}_{\mathbb{C}(s)}(s\widetilde{E}-\widetilde{A}) = l$, such that*

$$(s\mathcal{E}-\mathcal{A})V = W(s\widetilde{E}-\widetilde{A}). \tag{2.4}$$

In what follows we introduce special properties of matrix pencils $[sI-A\,,\,B] \in \mathbb{C}[s]^{n,n+m}$. In systems theory these properties correspond to trajectory design and stabilization of systems $\dot{x}(t) = Ax(t) + Bu(t)$ and are also known under the name *Hautus criteria*.

DEFINITION 2.5. *Let a pair $(A,B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ be given. Then we have the following:*

(i) *$\lambda \in \mathbb{C}$ is called an* uncontrollable mode *of (A,B) if it is a generalized eigenvalue of $[\,sI-A\,,\,B\,]$;*
(ii) *(A,B) is called* controllable *if it does not have any uncontrollable modes;*
(iii) *(A,B) is called* stabilizable *if all uncontrollable modes have negative real part.*

Finally, we present some notations about (possibly indefinite) inner products induced by a Hermitian matrix.

DEFINITION 2.6. *Let an Hermitian matrix $\mathcal{M} \in \mathbb{C}^{N,N}$ be given.*

(i) *Two subspaces $\mathcal{V}_1, \mathcal{V}_2 \subset \mathbb{C}^N$ are called* $\mathcal{M}$-orthogonal *if $\mathcal{V}_2 \subset \mathcal{V}_1^{\mathcal{M}\perp}$.*
(ii) *A subspace $\mathcal{V} \subset \mathbb{C}^N$ is called* $\mathcal{M}$-neutral *if $\mathcal{V}$ is $\mathcal{M}$-orthogonal to itself.*

2.3. Deflating subspaces and (neutral) Wong sequences. It is immediate that in the KCF (2.2) and EKCF (2.3), the space spanned by the columns of U_r (resp., U) that correspond to a single block defines a deflating subspace. Roughly speaking, we now give a characterization of these spaces without making use of the full KCF or EKCF. This is obtained by using the so-called Wong sequences [51, 10, 11].

The *Wong sequence* $(\mathcal{W}_\lambda^{(\ell)})$ of a pencil $s\mathcal{E}-\mathcal{A} \in \mathbb{C}[s]^{M,N}$ associated to a given $\lambda \in \mathbb{C}$ is the sequence of subspaces defined recursively by

$$\mathcal{W}_\lambda^{(0)} = \{0\}, \quad \mathcal{W}_\lambda^{(\ell)} = (\lambda\mathcal{E}-\mathcal{A})^{-1}(\mathcal{E}\mathcal{W}_\lambda^{(\ell-1)}), \quad \ell \in \mathbb{N}, \tag{2.5a}$$

while the Wong sequence for $\lambda = \infty$ is defined via

$$\mathcal{W}_\infty^{(0)} = \{0\}, \quad \mathcal{W}_\infty^{(\ell)} = \mathcal{E}^{-1}(\mathcal{A}\mathcal{W}_\infty^{(\ell-1)}), \quad \ell \in \mathbb{N}. \tag{2.5b}$$

It is shown in [51, 10, 11] that $(\mathcal{W}_\lambda^{(\ell)})$ is an increasing sequence of nested subspaces (i.e., $\mathcal{W}_\lambda^{(\ell-1)} \subseteq \mathcal{W}_\lambda^{(\ell)}$), and, by reasons of finite-dimensionality, we have stagnation of this sequence. We define

$$\mathcal{W}_\lambda := \bigcup_{\ell=0}^{\infty} \mathcal{W}_\lambda^{(\ell)}. \tag{2.6}$$

In the following, we show that $\mathcal{W}_\lambda$ is exactly the sum of the deflating subspaces associated to blocks corresponding to the generalized eigenvalue $\lambda \in \mathbb{C} \cup \{\infty\}$ together with the space corresponding to blocks of type K3.

First we present an auxiliary result stating that Wong sequences of a block-diagonal pencil are formed by direct sums of separate Wong sequences. It is furthermore shown how the pre- and postmultiplication of a pencil by invertible matrices influences Wong sequences.

Lemma 2.7. *Let $\lambda \in \mathbb{C} \cup \{\infty\}$ and a pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ be given.*

(i) *If $(\mathcal{W}_\lambda^{(\ell)})$ is a Wong sequence for $s\mathcal{E} - \mathcal{A}$, and $U_l \in \mathrm{Gl}_M(\mathbb{C})$, $U_r \in \mathrm{Gl}_N(\mathbb{C})$, then the corresponding Wong sequence for $U_l(s\mathcal{E}-\mathcal{A})U_r$ is given by $(U_r^{-1}\mathcal{W}_\lambda^{(\ell)})$.*

(ii) *Let $(\mathcal{W}_\lambda^{(\ell)})$, $(\widetilde{\mathcal{W}}_\lambda^{(\ell)})$ be Wong sequences for $s\mathcal{E} - \mathcal{A}$ and $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$, respectively. Then the corresponding Wong sequence for the pencil $s\,\mathrm{diag}(\mathcal{E}, \widetilde{\mathcal{E}}) - \mathrm{diag}(\mathcal{A}, \widetilde{\mathcal{A}})$ is given by $(\mathcal{W}_\lambda^{(\ell)} \times \widetilde{\mathcal{W}}_\lambda^{(\ell)})$.*

This enables us to consider the Wong sequences of the blocks in the KCF separately. It is easy to work out directly what happens on a single block of a Kronecker canonical form. For instance, for $\lambda = \infty$, direct computation shows that $\mathcal{W}_\infty^{(\ell)} = \{0\}$ for all ℓ on a K1 or K4 block, while for either a block of type K2 with size $k_j \times k_j$ or a block of type K3 with size $(k_j - 1) \times k_j$ we obtain that

$$\mathcal{W}_\infty^{(\ell)} = \begin{cases} \mathrm{span}\{e_1, \ldots, e_\ell\}, & \ell < k_j, \\ \mathbb{C}^{k_j}, & \ell \geq k_j. \end{cases}$$

As a consequence of these computations and Lemma 2.7, we can formulate the subsequent result that connects the subspace $\mathcal{W}_\lambda$ (which obviously does not depend on the particular choice of the matrices U_r and U_l as in (2.2)) to the space spanned by certain columns of U_r.

Corollary 2.8. *Let $\lambda \in \mathbb{C} \cup \{\infty\}$ and a pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ be given. Let $U_l \in \mathrm{Gl}_M(\mathbb{C})$, $U_r \in \mathrm{Gl}_N(\mathbb{C})$ such that $U_l(s\mathcal{E} - \mathcal{A})U_r$ is in KCF* (2.2)*. Further, let U_r be partitioned conformably with the KCF as*

$$U_r = \begin{bmatrix} U_1 & U_2 & \cdots & U_k \end{bmatrix}.$$

Then, for $\mathcal{W}_\lambda$ as in (2.6)*, there holds*

$$\mathcal{W}_\lambda = \sum_{j \in T_\lambda \cup S} \mathrm{im}\, U_j,$$

where

$$S = \{j \in \mathbb{N} \mid \mathcal{C}_j \text{ is of type } K3\},$$

$$T_\lambda = \begin{cases} \{j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type } K1 \text{ with eigenvalue } \lambda\} & \text{if } \lambda \in \mathbb{C}, \\ \{j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type } K3\} & \text{if } \lambda = \infty. \end{cases}$$

A further consequence is that, for any $\lambda, \mu \in \mathbb{C} \cup \{\infty\}$ with $\lambda \neq \mu$, there holds $\mathcal{W}_\lambda \cap \mathcal{W}_\mu = \sum_{j \in S} \operatorname{im} U_j$. Hence, the deflating subspace corresponding to all blocks of type K3 is well defined as well. We now present an auxiliary result on Wong sequences of upper triangular matrix pencils, which will be an essential ingredient for one of the main results of this article.

LEMMA 2.9. *Let the matrix pencils $s\mathcal{E}_{ij} - \mathcal{A}_{ij} \in \mathbb{C}[s]^{M_i,N_j}$ be given for $(i,j) \in \{(1,1),(1,2),(2,2)\}$. For $\lambda \in \mathbb{C} \cup \{\infty\}$, denote $\mathcal{W}_{\lambda,11}$ and $\mathcal{W}_\lambda$ to be the spaces at which the Wong sequences of the pencils $s\mathcal{E}_{11} - \mathcal{A}_{11}$ and, respectively,*

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} s\mathcal{E}_{11} - \mathcal{A}_{11} & s\mathcal{E}_{12} - \mathcal{A}_{12} \\ 0 & s\mathcal{E}_{22} - \mathcal{A}_{22} \end{bmatrix}$$

stagnate. Assume that the KCF of $s\mathcal{E}_{11} - \mathcal{A}_{11}$ does not contain any blocks of type $K4$ and, moreover, $\dim \mathcal{W}_{\lambda,11} = \dim \mathcal{W}_\lambda$. Then $\mathcal{W}_\lambda = \mathcal{W}_{\lambda,11} \times \{0\}$ and $\ker \lambda\mathcal{E}_{22} - \mathcal{A}_{22} = \{0\}$.

Proof. We show only the result for $\lambda \in \mathbb{C}$. The case of infinite eigenvalue can be proven by reversing the roles of $\mathcal{E}$ and $\mathcal{A}$, and then setting $\lambda = 0$.

By the upper triangularity of $s\mathcal{E} - \mathcal{A}$ and the construction of the Wong sequences, we immediately obtain that $\mathcal{W}_{\lambda,11} \times \{0\}$ is a subset of $\mathcal{W}_\lambda$. Since the dimensions of these spaces are equal, we obtain $\mathcal{W}_{\lambda,11} \times \{0\} = \mathcal{W}_\lambda$.

Using that the KCF of $s\mathcal{E}_{11} - \mathcal{A}_{11}$ does not contain any blocks of type K4, we may employ the KCF to obtain the identity

$$\mathcal{E}_{11}\mathcal{W}_{\lambda,11} + \operatorname{im}(\lambda\mathcal{E}_{11} - \mathcal{A}_{11}) = \mathbb{C}^{N_1}. \tag{2.7}$$

Now assume that $y \in \ker(\lambda\mathcal{E}_{22} - \mathcal{A}_{22})$. Then, by (2.7), there exists some $x \in \mathbb{C}^{N_1}$ with

$$(\lambda\mathcal{E}_{11} - \mathcal{A}_{11})x + (\lambda\mathcal{E}_{12} - \mathcal{A}_{12})y \in \mathcal{E}_{11}\mathcal{W}_{\lambda,11}.$$

Hence,

$$\begin{bmatrix} \lambda\mathcal{E}_{11} - \mathcal{A}_{11} & \lambda\mathcal{E}_{12} - \mathcal{A}_{12} \\ 0 & \lambda\mathcal{E}_{22} - \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{E}_{11}\mathcal{W}_{\lambda,11} \times \{0\},$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \lambda\mathcal{E}_{11} - \mathcal{A}_{11} & \lambda\mathcal{E}_{12} - \mathcal{A}_{12} \\ 0 & \lambda\mathcal{E}_{22} - \mathcal{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ 0 & \mathcal{E}_{22} \end{bmatrix} \cdot (\mathcal{W}_{\lambda,11} \times \{0\}) = \mathcal{W}_{\lambda,11} \times \{0\}.$$

However, this implies $y = 0$. Altogether, we have $\ker(\lambda\mathcal{E}_{22} - \mathcal{A}_{22}) = \{0\}$, whence λ is no generalized eigenvalue of $s\mathcal{E}_{22} - \mathcal{A}_{22}$. ∎

In what follows we extend the theory of Wong sequences to obtain $\mathcal{E}$-neutral deflating subspaces of even matrix pencils, which are essential for our theoretical and algorithmic framework for Lur'e equations. By a closer look at the EKCF (2.3), it can be realized that for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, the space $\mathcal{W}_\lambda$ is $\mathcal{E}$-neutral. However, this does not hold for imaginary or infinite generalized eigenvalues. The following modification of Wong sequences provides a suitable "$\mathcal{E}$-neutral part" of these subspaces. We define the *neutral Wong sequence* $(\mathcal{V}_{i\omega}^{(\ell)})$ associated with the imaginary eigenvalue $\lambda \in i\mathbb{R}$) via

$$\mathcal{Z}_\lambda^{(0)} = \mathcal{V}_\lambda^{(0)} = \{0\}, \tag{2.8a}$$

$$\mathcal{Z}_\lambda^{(\ell)} = (\lambda\mathcal{E} - \mathcal{A})^{-1}\left(\mathcal{E}\mathcal{V}_\lambda^{(\ell-1)}\right), \tag{2.8b}$$

$$\mathcal{V}_\lambda^{(\ell)} = \mathcal{V}_\lambda^{(\ell-1)} + \left(\mathcal{Z}_\lambda^{(\ell)} \cap \left(\mathcal{Z}_\lambda^{(\ell)}\right)^{\mathcal{E}\perp}\right), \qquad \ell \in \mathbb{N}, \tag{2.8c}$$

and the corresponding sequence for the infinite eigenvalue as

$$\mathcal{Z}_\infty^{(0)} = \mathcal{V}_\infty^{(0)} = \{0\}, \tag{2.8d}$$

$$\mathcal{Z}^{(\ell)} = \mathcal{E}^{-1}\left(\mathcal{A}\mathcal{V}_\infty^{(\ell-1)}\right), \tag{2.8e}$$

$$\mathcal{V}_\infty^{(\ell)} = \mathcal{V}_\infty^{(\ell-1)} + \left(\mathcal{Z}_\infty^{(\ell)} \cap \left(\mathcal{Z}_\infty^{(\ell)}\right)^{\mathcal{E}\perp}\right), \qquad \ell \in \mathbb{N}. \tag{2.8f}$$

It is obvious from its definition that $(\mathcal{V}_\lambda^{(\ell)})$ is an increasing and eventually stagnating sequence of nested subspaces, and we may define the subspace

$$\mathcal{V}_\lambda := \bigcup_{\ell=0}^{\infty} \mathcal{V}_\lambda^{(\ell)}. \tag{2.9}$$

Furthermore, if for the "conventional Wong sequence" $(W_\lambda^{(\ell)})$ there holds that $W_\lambda^{(\ell)}$ is $\mathcal{E}$-neutral for $\ell = 0, 1, 2, \dots, h$, then $\mathcal{V}_\lambda^{(\ell)} = \mathcal{W}_\lambda^{(\ell)}$ for $\ell = 1, 2, \dots, h$.

The following statement (which is analogous to Lemma 2.7) applies to $(\mathcal{V}_{i\omega}^{(\ell)})$ and shows that we may consider separately the blocks in the EKCF when analyzing the neutral Wong sequences.

LEMMA 2.10. *Let $\lambda \in i\mathbb{R} \cup \{\infty\}$ and an even matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$ be given.*

(i) *If $(\mathcal{V}_\lambda^{(\ell)})$ is a neutral Wong sequence for $s\mathcal{E} - \mathcal{A}$ and $U \in \mathrm{Gl}_N(\mathbb{C})$, then the corresponding neutral Wong sequence for $U^*(s\mathcal{E} - \mathcal{A})U$ is given by $(U^{-1}\mathcal{V}_\lambda^{(\ell)})$.*

(ii) *If $(\mathcal{V}_\lambda^\ell)$, $(\widetilde{\mathcal{V}}_\lambda^\ell)$ are neutral Wong sequences for $s\mathcal{E} - \mathcal{A}$ and $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$, respectively, then the corresponding neutral Wong sequence for $s\,\mathrm{diag}(\mathcal{E}, \widetilde{\mathcal{E}}) - \mathrm{diag}(\mathcal{A}, \widetilde{\mathcal{A}})$ is given by $(\mathcal{V}_\lambda^{(\ell)} \times \widetilde{\mathcal{V}}_\lambda^{(\ell)})$.*

Again, we can explicitly characterize the space at which neutral Wong sequences stagnate.

THEOREM 2.11. *Let $\mathcal{V}_\lambda^{(\ell)}$ be the neutral Wong sequence associated to $\lambda \in i\mathbb{R} \cup \{\infty\}$ for the even pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$. Let U be a nonsingular matrix reducing it to EKCF as in* (2.3)*, partitioned conformably as*

$$U = \begin{bmatrix} U_1 & U_2 & \cdots & U_k \end{bmatrix}.$$

Then, for $\mathcal{V}_\lambda$ as in (2.9)*, there holds*

$$\mathcal{V}_\lambda = \sum_{j \in T_\lambda} \mathrm{im}\left(U_j \begin{bmatrix} I_{h_j} \\ 0_{k_j - h_j, h_j} \end{bmatrix}\right) + \sum_{j \in S} \mathrm{im}\left(U_j \begin{bmatrix} I_{k_j} \\ 0_{k_j - 1, k_j} \end{bmatrix}\right),$$

where

$$S = \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type E4} \},$$

$$T_\lambda = \begin{cases} \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type E2 with eigenvalue } \lambda \} & \text{if } \lambda \in i\mathbb{R}, \\ \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type E3} \} & \text{if } \lambda = \infty, \end{cases}$$

$$h_j = \begin{cases} \lfloor \frac{k_j}{2} \rfloor & \text{if } \lambda \in i\mathbb{R}, \\ \lfloor \frac{k_j + 1}{2} \rfloor & \text{if } \lambda = \infty. \end{cases}$$

In particular, the subspaces $\mathcal{V}_\lambda$ are all $\mathcal{E}$-neutral and do not depend on the choice of U.

Proof. Lemma 2.10 allows us to restrict to the case where $s\mathcal{E} - \mathcal{A}$ is a single block of one of the four types in Table 2.2.

E1 Since $\lambda \in i\mathbb{R} \cup \{\infty\}$, both matrices $\mathcal{E}$, $\lambda\mathcal{E} - \mathcal{A}$ are nonsingular, whence $\mathcal{V}_\lambda^{(\ell)} = \{0\}$.

E2 $\mathcal{V}_\lambda^{(\ell)} = \{0\}$ unless λ coincides with the generalized eigenvalue associated to given block. It therefore suffices to consider only the latter case. Explicit computation shows that $\mathcal{V}_\lambda^{(\ell)} = \operatorname{span}\{e_1, e_2, \ldots, e_\ell\}$ for $\ell \leq h_j$. After that, $\mathcal{Z}_\lambda^{(h_j+1)}$ is not $\mathcal{E}$-neutral anymore. The computations in the case of even and odd k_j slightly differ, but in both we obtain $(\mathcal{Z}_\lambda^{(h_j+1)})^{\mathcal{E}\perp} \subseteq \mathcal{V}_\lambda^{(h_j)}$, thus $\mathcal{V}_\lambda^{(h_j+1)} = \mathcal{V}_\lambda^{(h_j)}$, and the sequence stagnates.

E3 Here we have $\mathcal{V}_\lambda^{(\ell)} = \{0\}$ unless $\lambda = \infty$, so we consider only this case: However, a similar argumentation to that described for the case of a block of type E2 can be applied here to obtain the desired result.

E4 A block of type E4 is antidiagonally composed of the block of type K3 and a block of type K4. For the latter, the "conventional Wong sequence" is trivial, i.e., $\mathcal{W}_\lambda^{(\ell)} = \{0\}$; for the former, the conventional Wong sequence reaches $\mathbb{C}^{k_j}$ after k_j steps. Therefore, for any $\lambda \in i\mathbb{R} \cup \{\infty\}$ the Wong sequence $\mathcal{W}_\lambda^{(\ell)}$ of an E4 block fulfills $\mathcal{W}_\lambda^{(k_j)} = \operatorname{span}\{e_1, e_2, \ldots, e_{k_j}\}$. Since this subspace is $\mathcal{E}$-neutral, we may apply the statement that for the conventional and neutral Wong sequences there holds $\mathcal{W}_\lambda^{(\ell)} = \mathcal{V}_\lambda^{(\ell)}$ if $\mathcal{W}_\lambda^{(\ell)}$ is $\mathcal{E}$-neutral. Hence, we have $\mathcal{V}_\lambda = \mathcal{W}_\lambda = \operatorname{span}\{e_1, e_2, \ldots, e_{k_j}\}$. ◻

3. Lur'e equations and deflating subspaces of even matrix pencils.

Solvability and structure of the solution set of the Lur'e equations (1.1) are described in [40]. In particular, the eigenstructure of the associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ (1.9) can be related to solutions of (1.1) in a way that these define deflating subspaces via

$$\begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -X & K^*JV \\ 0 & L^*JV \end{bmatrix} \begin{bmatrix} -sI + A & B \\ V^*K & V^*L \end{bmatrix}, \tag{3.1}$$

where $V \in \mathbb{C}^{p,m}$ is a matrix with $V^*V = I_p$ and $\operatorname{im}(V) = \operatorname{im}([K\,,\, L])$. The property $X = X^*$ is equivalent to this deflating subspace being $\mathcal{E}$-neutral.

Definition 3.1. *Let $A, Q \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, and $R \in \mathbb{R}^{m,m}$ be given with $Q = Q^*$, $R = R^*$. Further, let $J \in \mathbb{R}^{m,m}$ be a signature matrix. Then a solution $X \in \mathbb{C}^{n,n}$ of the Lur'e equations* (1.1) *with $X = X^*$ is called* stabilizing (antistabilizing) *if*

$$\operatorname{rank} \begin{bmatrix} -\lambda I + A & B \\ K & L \end{bmatrix} = n + p \quad \textit{for all } \lambda \in \mathbb{C}^+ \ (\lambda \in \mathbb{C}^-). \tag{3.2}$$

Condition (3.2) is equivalent to the KCF of the pencil

$$\begin{bmatrix} -sI + A & B \\ K & L \end{bmatrix}$$

having the following properties: All blocks of type K4 are of size 1×0 and, moreover, all generalized eigenvalues having nonpositive (nonnegative) real part. Note that in the case of invertible R the concept of (anti-) stabilizing solution introduced above coincides with the corresponding notion for algebraic Riccati equations [32]. For the sake of brevity and analogy, we mainly focus on stabilizing solutions in this article.

Remark 3.2 (stabilizing solutions).

(a) It is shown in [40] that, in the case $J = I_m$, a stabilizing solution X is maximal, where the word "maximal" has to be understood in terms of definiteness. More precisely, all other solutions Y of the Lur'e equations fulfill $X \geq Y$. In an analogous way, antistabilizing solutions are minimal with respect to definiteness.

(b) If $J = -I_m$, a statement reverse to (a) holds true: The stabilizing solution is minimal; the antistabilizing solution is maximal with respect to definiteness. This holds true since we can lead back this setup to the case $J = I_m$ by a simple substitution $(X, Q, S, R) \rightsquigarrow (-X, -Q, -S, -R)$.

(c) If J is indefinite, there is no relation between extremality and (anti-) stabilizability of solutions.

As we have seen in (3.1), solutions to Lur'e equations define $\mathcal{E}$-neutral deflating subspaces of the even matrix pencil (1.9). It is shown in [40] that also the converse holds true; that is, the solutions of the Lur'e equations can be constructed from certain $\mathcal{E}$-neutral deflating subspaces of $s\mathcal{E} - \mathcal{A}$. First we collect several necessary and sufficient criteria for the existence of a stabilizing solution of the Lur'e equations (1.1). In particular, criteria in terms of the EKCF of (1.9) are presented. Thereby we need the following two criteria:

P1 All blocks of type E2 in the EWCF of $s\mathcal{E} - \mathcal{A}$ have even size.

P2 All blocks of type E3 in the EWCF of $s\mathcal{E} - \mathcal{A}$ have odd size.

THEOREM 3.3. *Let $A, Q \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, and $R \in \mathbb{R}^{m,m}$ be given with $Q = Q^*$, $R = R^*$. Further, let $J \in \mathbb{R}^{m,m}$ be a signature matrix. Let the pencil $s\mathcal{E} - \mathcal{A}$ be as in* (1.9)*, let the spectral density function $\Phi(s) \in \mathbb{C}(s)^{m,m}$ be as in* (1.3)*. Then the following holds true:*

(a) *If a solution of the Lur'e equations* (1.1) *exists, then for all $\omega \in \mathbb{R}$ such that $i\omega$ is not an eigenvalue of A, the spectral density function fulfills $n_+(\Phi(i\omega)) \leq n_+(J)$ and $n_-(\Phi(i\omega)) \leq n_-(J)$.*

(b) *If a solution of the Lur'e equations* (1.1) *exists, then $s\mathcal{E} - \mathcal{A}$ fulfills* P1 *and* P2. *Furthermore, the number of blocks of type E*3 *with positive (negative) signature does not exceed* $\max_{\omega \in \mathbb{R} \setminus \{-i\sigma(A)\}} n_+(\Phi(i\omega))$ *(resp.,* $\max_{\omega \in \mathbb{R} \setminus \{-i\sigma(A)\}} n_-(\Phi(i\omega))$*).*

(c) *If a stabilizing solution of the Lur'e equations* (1.1) *exists, then* (A, B) *is stabilizable.*

(d) *Assume that $J = I_m$ ($J = -I_m$): If* P1 *and* P2 *hold true, then all blocks of type E*3 *in the EWCF of $s\mathcal{E} - \mathcal{A}$ have positive (negative) sign and, moreover, at least one of the properties*
   (i) *the pair (A, B) is stabilizable and the pencil $s\mathcal{E} - \mathcal{A}$ as in* (1.9) *is regular;*
   (ii) *the pair (A, B) is controllable;*

is fulfilled, then a stabilizing solution of the Lur'e equations (1.1) *exists.*

Proof. Statement (a) follows from a brief verification of (1.2) and (1.4). The first part of assertion (b) is shown in [3]. The signature conditions in (b) can be derived from [18, Lemma 5.2]. Condition (c) follows immediately from the definition of the stabilizing solution in (3.2). Criterion (d) has been shown, for $J = I_m$, in [40]. The case of negative definite J can be proven analogously. ∎

Remark 3.4 (solvability of Lur'e equations).

(a) The work [3] presents a further criterion that is sufficient for the existence of both a stabilizing solution and an antistabilizing solution: In the case where R is invertible, these exist, if for all $x \in \mathbb{C}^n$ holds

$$x^*(sI-A)^{-1}B(R+B^*(-sI-A^*)^{-1}Q(sI-A)^{-1}B)B^*(-sI-A^*)^{-1} \in \mathbb{R}(s)\backslash\{0\}.$$

Note that, if R is positive definite, this condition reduces to controllability of (A, B).

(b) In the case where $J \in \{\pm I_m\}$, solvability of the Lur'e equations can be characterized by the feasibility of a linear matrix inequality [3].

(c) It follows from (3.1) that for $p = \operatorname{rank}[K\ L]$, the number of blocks of type E4 in the EKCF of $s\mathcal{E} - \mathcal{A}$ equals $m - p$. If, additionally, there holds $\operatorname{rank} R = p$ and $J \in \{\pm I_m\}$, there exists a unitary transformation with

$$U_1^* R U_1 = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2^* L U_1 = \begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2^* K = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $L_1, R_1 \in \mathbb{C}^{p,p}$ are invertible and $K \in \mathbb{C}^{p,n}$. By partitioning $BU_1 = [B_1\ ,\ B_2]$, $SU_1 = [S_1\ ,\ S_2]$, the Lur'e equations are now equivalent to

$$\begin{aligned} &A^*X + XA + Q \pm (XB_1 + S_1)R_1^{-1}(XB_1 + S_1)^* = 0, \\ &XB_2 + S_2 = 0. \end{aligned}$$

This is equivalent to the constrained Riccati equations (1.7) that originally have been introduced in [26, 48]. Note that the above approach is not possible if $\operatorname{rank} R < p$, which is equivalent to the EKFC containing blocks of type E3. This is in accordance with the solvability criteria for constrained Riccati equations in [26, 48]: For an *extended Hamiltonian matrix pencil* (which can be transformed to the even pencil (1.9) via simple row transformations), solvability of (1.7) has been related to the absence of infinite eigenvalues.

A simple example that illustrates that Lur'e equations treat a more general case is the following: For $n = m = 1$, consider $A = Q = B = S = 1$, $J = -1$ and $R = 0$. The Lur'e equations then read $2X + 1 = -K^2$, $X + 1 = -KL$, $0 = -L^2$. These have the unique solution $X = -1$. Using $R^+ = 0$, the constrained Riccati equation reads $2X + 1 = 0$, $X + 1 = 0$. The latter system is however unsolvable.

The stabilizing solution can be explicitly constructed from deflating subspaces of the even matrix pencil (1.9): It is shown in [40, 3] that the *extended graph space*

$$\mathcal{G}_X = \operatorname{im} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} \tag{3.3a}$$

of the stabilizing solution X fulfills $\mathcal{G}_X = \mathcal{V}_s$, where

$$\mathcal{V}_s = \left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \right) + \mathcal{V}_\infty. \tag{3.3b}$$

In other words, for matrices $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$ with

$$\operatorname{im} \begin{bmatrix} V_\mu \\ V_x \\ V_u \end{bmatrix} = \mathcal{V}_s, \tag{3.3c}$$

there holds $X = V_\mu V_x^-$, where $V_x^- \in \mathbb{C}^{n+m,n}$ is an arbitrary right inverse of V_x, i.e., $V_x^- V_x = I$. Moreover, we have $\operatorname{rank}_{\mathbb{C}(s)}(s\mathcal{E} - \mathcal{A}) = \operatorname{rank}[K\,,\,L] + 2n$. Besides being crucial for all of our numerical considerations in this paper, the correspondence (3.3) also provides us an equivalent criterion for the solvability of the Lur'e equations (1.1).

THEOREM 3.5 (see [3, 40]). *Let $A, Q \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, and $R \in \mathbb{R}^{m,m}$ be given with $Q = Q^*$, $R = R^*$. Further, let $J \in \mathbb{R}^{m,m}$ be a signature matrix, and let $\mathcal{V}_s$ be defined as in* (3.3b). *Then a stabilizing solution of the Lur'e equations* (1.1) *exists if and only if* $\dim \mathcal{V}_s = n + m$ *and for matrices $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$ with* (3.3c). *Then a stabilizing solution exists if and only if* $\operatorname{rank} V_\mu = n$. *In this case, there holds $X = V_\mu V_x^-$.*

Remark 3.6. In the case where the matrices A, Q, B, S, and R are all real, then the space $\mathcal{V}_\infty$ is real, too. Since the spaces $\mathcal{W}_\lambda + \mathcal{W}_{\overline{\lambda}}$ and $\mathcal{V}_\mu + \mathcal{V}_{\overline{\mu}}$ are real as well for any generalized eigenvalues $\lambda \in \mathbb{C}^-$, $\mu \in i\mathbb{R}$, it can be verified that the stabilizing solution is real in this case. Note that all numerical algorithms that will be introduced in this paper avoid complex arithmetic if A, B, S, Q, and R are all real.

The following result is a direct conclusion from the relations in (3.3). It is shown that the stabilizing solution of the Lur'e equations satisfies a certain identity with the matrices generating some deflating subspace $\breve{\mathcal{V}}$ of $s\mathcal{E} - \mathcal{A}$ with

$$\breve{\mathcal{V}} \subset \mathcal{V}_s. \tag{3.4}$$

COROLLARY 3.7. *Let $A, Q \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, and $R \in \mathbb{R}^{m,m}$ be given with $Q = Q^*$, $R = R^*$. Further, let $J \in \mathbb{R}^{m,m}$ be a signature matrix. Assume that the Lur'e equations* (1.1) *have a stabilizing solution. Let $\breve{\mathcal{V}}$ be an r-dimensional deflating subspace of $s\mathcal{E} - \mathcal{A}$ such that* (3.4) *holds true. Then, for $\breve{V}_\mu$, $\breve{V}_x \in \mathbb{C}^{2n+m,r}$, $\breve{V}_u \in \mathbb{C}^{2n+m,r}$ with*

$$\breve{\mathcal{V}} = \operatorname{im} \begin{bmatrix} \breve{V}_\mu \\ \breve{V}_x \\ \breve{V}_u \end{bmatrix}, \tag{3.5}$$

there holds $\ker \breve{V}_x \subset \ker \breve{V}_\mu$. *Moreover, the stabilizing solution X of* (1.1) *satisfies* $X\breve{V}_x = \breve{V}_\mu$.

Remark 3.8. Note that for any deflating subspace $\breve{\mathcal{V}}$ with (3.4), the space

$$\breve{\mathcal{V}} + (\{0\} \times \{0\} \times \mathbb{C}^m)$$

is an $\mathcal{E}$-neutral deflating subspace which is also a subset of G_X. Hence we can make use of Corollary 3.7 to see that it is no loss of generality to assume that

$$\breve{\mathcal{V}} = \operatorname{im} \underbrace{\begin{bmatrix} \breve{V}_\mu & 0 \\ \breve{V}_x & 0 \\ 0 & I_m \end{bmatrix}}_{=:\breve{V}}. \tag{3.6}$$

Furthermore, $\breve{V}$ has full column rank if and only if $\breve{V}_x$ has full column rank. Therefore, we may assume in the following that $\breve{V}_x$ has a left inverse $\breve{V}_x^-$, i.e., the relation $\breve{V}_x^- \breve{V}_x = I$ holds true.

3.1. Projected Lur'e equations. Now we extend some of the terminology and solution theory to projected Lur'e equations (1.11) with (1.12). These equations will occur in later parts after a certain transformation of standard Lur'e equations.

In theory, we may change coordinates so that the equations are equivalent to a system of Lur'e equations of smaller dimension. Namely, for $T \in \mathrm{Gl}_n(\mathbb{C})$ with

$$\Pi = T\,\mathrm{diag}(I,0)T^{-1}, \tag{3.7a}$$

conditions (1.12) imply

$$T^{-1}\widetilde{A}T = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad T^*\widetilde{X}T = \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad T^*\widetilde{Q}T = \begin{bmatrix} \widetilde{Q}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.7b}$$

$$T^{-1}\widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ 0 \end{bmatrix}, \quad T^*\widetilde{S} = \begin{bmatrix} \widetilde{S}_1 \\ 0 \end{bmatrix}, \tag{3.7c}$$

and we are led back to Lur'e equations in standard form

$$\begin{aligned} \widetilde{A}_{11}^*\widetilde{X}_{11} + \widetilde{X}_{11}\widetilde{A}_{11} + \widetilde{Q}_{11} &= \widetilde{K}_1^* J \widetilde{K}_1, \\ \widetilde{X}_{11}\widetilde{B}_1 + \widetilde{S}_1 &= \widetilde{K}_1^* J \widetilde{L}, \\ \widetilde{R} &= \widetilde{L}^* J \widetilde{L}. \end{aligned} \tag{3.8}$$

In practice, we would like to avoid this transformation for numerical reasons.

Definition 3.9. *We say that $\widetilde{X}$ is a (stabilizing) solution of the projected Lur'e equations* (1.11) *if there holds*

$$\mathrm{im}\begin{bmatrix} \widetilde{X} & 0 \\ \Pi & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Pi^* & 0 & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \left(\left(\sum_{\lambda \in \mathbb{C}^-} \widetilde{\mathcal{W}}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \widetilde{\mathcal{V}}_\lambda \right) + \widetilde{\mathcal{V}}_\infty \right), \tag{3.9}$$

where $\widetilde{\mathcal{W}}_\lambda$, $\widetilde{\mathcal{V}}_\lambda$, and $\widetilde{\mathcal{V}}_\infty$ are the corresponding spaces obtained by the (neutral) Wong sequences of the even pencil

$$s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}} = \begin{bmatrix} 0 & -s\Pi + \widetilde{A} & \widetilde{B} \\ s\Pi^* + \widetilde{A}^* & \widetilde{Q} & \widetilde{S} \\ \widetilde{B}^* & \widetilde{S}^* & \widetilde{R} \end{bmatrix}. \tag{3.10}$$

It follows immediately that $\widetilde{X}$ is the stabilizing solution of the projected Lur'e equations (1.11) with (1.12) if and only if $\widetilde{X}_{11}$ with (3.7) is the stabilizing solution of the reduced Lur'e equations (3.8). As a consequence, Theorem 3.3 can be suitably generalized to the projected case. In particular, solvability of the projected Lur'e equations (1.11) with (1.12) implies that the pencil $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ as in (1.9) fulfills P1 and P2.

4. Partial solutions and projected Lur'e equations. If we have computed some deflating subspace $\check{\mathcal{V}} = \mathrm{im}\,\check{V} \subset G_X$ for some matrix $\check{V}$ as in (3.6) with full column rank, then Corollary 3.7 provides information on the action of X on a certain subspace.

In this section, we show that the "remaining part" of the stabilizing solution X solves projected Lur'e equations. As explained in Remark 3.8, we may assume that for $\check{V}$ as in (3.6), the submatrix $\check{V}_x \in \mathbb{C}^{n,\check{n}}$ possesses a left inverse $\check{V}_x^- \in \mathbb{C}^{\check{n},n}$. The matrix

$$\Pi = I_n - \check{V}_x\check{V}_x^- \in \mathbb{C}^{n,n} \tag{4.1}$$

is therefore a projector along $\operatorname{im}\check{V}_x$. Expanding the stabilizing solution X of the Lur'e equations (1.1) as

$$X = \Pi^* X \Pi + (I-\Pi)^* X + \Pi^* X (I-\Pi),$$

the relation $X(I-\Pi) = X\check{V}_x\check{V}_x^- = \check{V}_\mu \check{V}_x^-$ gives rise to

$$\begin{aligned} X &= \Pi^* X\Pi + (\check{V}_x^-)^*\check{V}_\mu^* + \Pi^*\check{V}_\mu\check{V}_x^- \\ &= \Pi^* X\Pi + (\check{V}_x^-)^*\check{V}_\mu^* + \check{V}_\mu\check{V}_x^- - (\check{V}_x^-)^*\check{V}_x^*\check{V}_\mu\check{V}_x^-. \end{aligned} \tag{4.2}$$

As a consequence, the problem of solving the Lur'e equations for X is reduced to the problem of solving for X on a subspace complementary to $\operatorname{im}\check{V}_x$. We therefore speak of *partially solving* the Lur'e equations. We describe in what follows that the matrix $\Pi^* X\Pi$ is indeed a solution of certain projected Lur'e equations (1.11).

Multiplying $A^*X + XA + Q = K^*JK$ (a) from the left with Π^* and from the right with Π, (b) from the left with $\check{V}_x^*$ and from the right with Π, and (c) from the left with $\check{V}_x^*$ and from the right with $\check{V}_x$ yields

(4.3a)
$$\Pi^*A^*\Pi^*X\Pi + \Pi^*X\Pi A\Pi + \Pi^*(A^*(\check{V}_x^-)^*\check{V}_\mu^* + \check{V}_\mu\check{V}_x^- A + Q)\Pi = (K\Pi)^*J(K\Pi),$$

$$\check{V}_x^*A^*\Pi^*X\Pi + \check{V}_x^*A^*(\check{V}_x^-)^*\check{V}_\mu^*\Pi + \check{V}_\mu^*A\Pi + V_x^*Q\Pi = (K\check{V}_x)^*J(K\Pi), \tag{4.3b}$$

$$\check{V}_x^*A^*\check{V}_\mu + \check{V}_\mu^*A\check{V}_x + \check{V}_x^*Q\check{V}_x = (K\check{V}_x)^*J(K\check{V}_x). \tag{4.3c}$$

Furthermore, a multiplication of $B^*X + S^* = L^*JK$ from the right with (a) Π and (b) $\check{V}_x$ gives

$$B^*\Pi^*X\Pi + (B^*(\check{V}_x^-)^*\check{V}_\mu^* + S^*)\Pi = L^*J(K\Pi), \tag{4.4a}$$

$$B^*\check{V}_\mu + S^*\check{V}_x = L^*J(K\check{V}_x). \tag{4.4b}$$

Then (4.3) and (4.4) imply that by setting

$$\widetilde{K} = K\Pi, \qquad \widetilde{L} = \begin{bmatrix} K\check{V}_x & L \end{bmatrix},$$

then $\widetilde{X} = \Pi^*X\Pi$ fulfills the projected Lur'e equation (1.11) with matrices

$$\begin{aligned} \widetilde{A} &= \Pi A\Pi, \\ \widetilde{Q} &= \Pi^*(A^*(\check{V}_x^-)^*\check{V}_\mu^* + \check{V}_\mu\check{V}_x^- A + Q)\Pi, \\ \widetilde{B} &= \begin{bmatrix} \Pi A\check{V}_x & \Pi B \end{bmatrix}, \\ \widetilde{S} &= \begin{bmatrix} \Pi^*\check{V}_\mu\check{V}_x^- A\check{V}_x + \Pi^*A^*\check{V}_\mu + \Pi^*QV_x & \Pi^*(S + \check{V}_\mu\check{V}_x^- B) \end{bmatrix}, \\ \widetilde{R} &= \begin{bmatrix} \check{V}_x^*A^*\check{V}_\mu + \check{V}_\mu^*A\check{V}_x + \check{V}_x^*Q\check{V}_x & \check{V}_\mu^*B + \check{V}_x^*S \\ B^*\check{V}_\mu + S^*\check{V}_x & R \end{bmatrix}. \end{aligned} \tag{4.5}$$

Conversely, the above computations imply that if $\widetilde{X}$ solves the projected Lur'e equations, then X as in (4.2) solves the original Lur'e equations (1.1). In particular, there holds $\widetilde{p} = p$.

In the following, we show that this reduction also preserves the property of $\widetilde{X} = \Pi^*X\Pi$ being stabilizing.

THEOREM 4.1. *Let $A, Q \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$, and $R \in \mathbb{R}^{m,m}$ be given with $Q = Q^*$, $R = R^*$. Further, let $J \in \mathbb{R}^{m,m}$ be a signature matrix. Assume that the Lur'e equations* (1.1) *have a stabilizing solution. Let $\breve{\mathcal{V}} \subset \mathbb{C}^{2n+m}$ be a deflating subspace of the even matrix pencil* (1.9) *with*

$$\{0\} \times \{0\} \times \mathbb{C}^m \subset \breve{\mathcal{V}} \subset \left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda\right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda\right) + \mathcal{V}_\infty$$

and

$$\breve{\mathcal{V}} = \operatorname{im} \breve{V} = \operatorname{im} \begin{bmatrix} \breve{V}_\mu & 0 \\ \breve{V}_x & 0 \\ 0 & I_m \end{bmatrix} \tag{4.6}$$

for some $\breve{V}_x \in \mathbb{C}^{n,\breve{n}}$ with full column rank. Let $\Pi = I_n - \breve{V}_x \breve{V}_x^-$, where $\breve{V}_x^- \in \mathbb{C}^{\breve{n},n}$ fulfills $\breve{V}_x^- \breve{V}_x = I_{\breve{n}}$. Then X is the stabilizing solution of the Lur'e equations (1.1) *if and only if $\widetilde{X} = \Pi^* X \Pi$ is a stabilizing solution of the projected Lur'e equations* (1.11) *with matrices as in* (4.5).

Proof. By the definition of deflating subspace, there exists a matrix $W \in \mathbb{C}^{2n+m,k}$ with $\operatorname{rank} W = k$ and a pencil $s\breve{E} - \breve{A} \in \mathbb{C}[s]^{k,\breve{n}+m}$ with $\operatorname{rank}_{\mathbb{C}(s)}(s\breve{E} - \breve{A}) = k$ and $(s\mathcal{E} - \mathcal{A})\breve{V} = \breve{W}(-s\breve{E} + \breve{A})$. In particular, the equation

$$\begin{bmatrix} -\breve{V}_x & 0 \\ \breve{V}_\mu & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{E}\breve{V} = \breve{W}\breve{E}$$

implies that $\breve{E} = \begin{bmatrix} \breve{E}_1 & 0_{k,m} \end{bmatrix}$ for some $\breve{E}_1 \in \mathbb{C}^{k,\breve{n}}$ with $\operatorname{rank} \breve{E}_1 = \breve{n}$. By a suitable change of coordinates in W, we can therefore assume that

$$-s\breve{E} - \breve{A} = \begin{bmatrix} -sI_{\breve{n}} + \breve{A}_{11} & \breve{A}_{12} \\ \breve{A}_{21} & \breve{A}_{22} \end{bmatrix},$$

and thereby, for some for some matrices $\breve{W}_{12}, \breve{W}_{22} \in \mathbb{C}^{n,k-\breve{n}}$, $\breve{W}_{32} \in \mathbb{C}^{m,k-\breve{n}}$,

$$W = \begin{bmatrix} \breve{V}_x & \breve{W}_{12} \\ -\breve{V}_\mu & \breve{W}_{22} \\ 0 & \breve{W}_{32} \end{bmatrix}.$$

Let $T_x \in \mathbb{C}^{n,n-\breve{n}}$ with $T_x = \Pi T_x \in \mathbb{C}^{n,n-\breve{n}}$. Then $\begin{bmatrix} \breve{V}_x & T_x \end{bmatrix}$ is a nonsingular (square) matrix and

$$\operatorname{im} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} = \operatorname{im} \begin{bmatrix} \breve{V}_\mu & X\Pi & 0 \\ \breve{V}_x & \Pi & 0 \\ 0 & 0 & I_m \end{bmatrix} = \operatorname{im} \begin{bmatrix} \breve{V}_\mu & XT_x & 0 \\ \breve{V}_x & T_x & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

and the rightmost matrix has full column rank. Hence there exist matrices $W_{13}, W_{23} \in \mathbb{C}^{n,n+p-k}$, $W_{33} \in \mathbb{C}^{n,n+p-k}$, and $E_{13}, A_{13} \in \mathbb{C}^{\breve{n},n-\breve{n}}$, $E_{23}, A_{23} \in \mathbb{C}^{k-\breve{n},n-\breve{n}}$, and $E_{33}, A_{33} \in \mathbb{C}^{n+p-k,n-\breve{n}}$ with

$$\begin{bmatrix} 0 & -sI+A & B \\ sI+A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \underbrace{\begin{bmatrix} \breve{V}_\mu & 0 & XT_x \\ \breve{V}_x & 0 & T_x \\ 0 & I_m & 0 \end{bmatrix}}_{=:V\in\mathbb{C}^{2n+m,m+n}}$$

$$= \underbrace{\begin{bmatrix} \breve{V}_x & \breve{W}_{12} & W_{13} \\ -\breve{V}_\mu & \breve{W}_{22} & W_{23} \\ 0 & \breve{W}_{32} & W_{33} \end{bmatrix}}_{=:W\in\mathbb{C}^{2n+m,n+p}} \underbrace{\begin{bmatrix} -sI_{\breve{n}}+\breve{A}_{11} & \breve{A}_{12} & -sE_{13}+A_{13} \\ \breve{A}_{21} & \breve{A}_{22} & -sE_{23}+A_{23} \\ 0 & 0 & -sE_{33}+A_{33} \end{bmatrix}}_{=:-s\tilde{E}+\tilde{A}\in\mathbb{C}[s]^{n+p,m+n}}.$$

The solution X is stabilizing if and only if $-\lambda\tilde{E}+\tilde{A}$ has full column rank for all $\lambda\in\mathbb{C}^+$. Due to our choice of the subspace $\breve{\mathcal{V}}$, this holds if and only if $-\lambda E_{33}+A_{33}$ has full row rank for all $\lambda\in\mathbb{C}^+$. Now consider the matrices

$$M_{\breve{\mathcal{V}}} = \begin{bmatrix} \Pi^* & (\breve{V}_x^-)^*\breve{V}_\mu^*\Pi & \breve{V}_\mu & 0 & (\breve{V}_x^-)^* \\ 0 & \Pi & \breve{V}_x & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \end{bmatrix}, \quad M_{\breve{\mathcal{V}}}^- = \begin{bmatrix} \Pi^* & -\Pi^*\breve{V}_\mu\breve{V}_x^- & 0 \\ 0 & \Pi & 0 \\ 0 & \breve{V}_x^- & 0 \\ 0 & 0 & I_m \\ \breve{V}_x^* & -\breve{V}_\mu^* & 0 \end{bmatrix}.$$

Then we have $M_{\breve{\mathcal{V}}}M_{\breve{\mathcal{V}}}^- = I$ and

$$M_{\breve{\mathcal{V}}}^*(s\mathcal{E}-\mathcal{A})M_{\breve{\mathcal{V}}} = \begin{bmatrix} 0 & -s\Pi+\widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 & 0 \\ s\Pi^*+\widetilde{A}^* & \widetilde{Q} & \widetilde{S}_1 & \widetilde{S}_2 & \widetilde{M}_1 \\ \widetilde{B}_1^* & \widetilde{S}_1^* & \widetilde{R}_{11} & \widetilde{R}_{12} & \widetilde{M}_2 \\ \widetilde{B}_2^* & \widetilde{S}_2^* & \widetilde{R}_{12}^* & \widetilde{R}_{22} & \widetilde{M}_3 \\ 0 & \widetilde{M}_1^* & \widetilde{M}_2^* & \widetilde{M}_3^* & 0 \end{bmatrix},$$

with $\widetilde{A}$ and $\widetilde{Q}$ as in (4.5),

$$\widetilde{M}_1 = \Pi^*AV_x^-, \quad \widetilde{M}_2 = sI+\breve{V}_x^*A\breve{V}_x, \quad \widetilde{M}_3 = B^*(\breve{V}_x)^*,$$

and

$$\widetilde{B} = \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 \end{bmatrix}, \quad \widetilde{S} = \begin{bmatrix} \widetilde{S}_1 & \widetilde{S}_2 \end{bmatrix}, \quad \widetilde{R} = \begin{bmatrix} \widetilde{R}_{11} & \widetilde{R}_{12} \\ \widetilde{R}_{12}^* & \widetilde{R}_{22} \end{bmatrix}.$$

Then an evaluation of the matrix products in

$$(M_{\breve{\mathcal{V}}}^*(s\mathcal{E}-\mathcal{A})M_{\breve{\mathcal{V}}})\cdot(M_{\breve{\mathcal{V}}}^-V) = (M_{\breve{\mathcal{V}}}^*W)\cdot(-s\tilde{E}+\tilde{A})$$

leads to

$$\begin{bmatrix} 0 & -s\Pi+\widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 & 0 \\ s\Pi^*+\widetilde{A}^* & \widetilde{Q} & \widetilde{S}_1 & \widetilde{S}_2 & \widetilde{M}_1 \\ \widetilde{B}_1^* & \widetilde{S}_1^* & \widetilde{R}_{11} & \widetilde{R}_{12} & \widetilde{M}_2 \\ \widetilde{B}_2^* & \widetilde{S}_2^* & \widetilde{R}_{12}^* & \widetilde{R}_{22} & \widetilde{M}_3 \\ 0 & \widetilde{M}_1^* & \widetilde{M}_2^* & \widetilde{M}_3^* & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \Pi^*XT_x \\ 0 & 0 & T_x \\ I_{\breve{n}} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \tilde{W}_{12} & \tilde{W}_{13} \\ 0 & \tilde{W}_{22} & \tilde{W}_{23} \\ 0 & \tilde{W}_{32} & \tilde{W}_{33} \\ 0 & \tilde{W}_{42} & \tilde{W}_{43} \\ I_{\breve{n}} & \tilde{W}_{52} & \tilde{W}_{53} \end{bmatrix} \cdot \begin{bmatrix} -sI_{\breve{n}}+\breve{A}_{11} & \breve{A}_{12} & -sE_{13}+A_{13} \\ \breve{A}_{21} & \breve{A}_{22} & -sE_{23}+A_{23} \\ 0 & 0 & -sE_{33}+A_{33} \end{bmatrix}$$

for some matrices $\tilde{W}_{ij}$ of suitable dimensions. Canceling the last row of this equation and, furthermore, realizing that the last block column of the matrix pencil on the left hand side has no influence on the product, we obtain

$$\begin{aligned}
&\begin{bmatrix} 0 & -s\Pi + \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ s\Pi^* + \widetilde{A}^* & \widetilde{Q} & \widetilde{S}_1 & \widetilde{S}_2 \\ \widetilde{B}_1^* & \widetilde{S}_1^* & \widetilde{R}_{11} & \widetilde{R}_{12} \\ \widetilde{B}_2^* & \widetilde{S}_2^* & \widetilde{R}_{12}^* & \widetilde{R}_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \Pi^* X T_x \\ 0 & 0 & T_x \\ I_{\breve{n}} & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \tilde{W}_{12} & \tilde{W}_{13} \\ 0 & \tilde{W}_{22} & \tilde{W}_{23} \\ 0 & \tilde{W}_{32} & \tilde{W}_{33} \\ 0 & \tilde{W}_{42} & \tilde{W}_{43} \end{bmatrix} \cdot \begin{bmatrix} -sI_{\breve{n}} + \breve{A}_{11} & \breve{A}_{12} & -sE_{13} + A_{13} \\ \breve{A}_{21} & \breve{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix} \qquad (4.7) \\
&= \begin{bmatrix} \tilde{W}_{12} & \tilde{W}_{13} \\ \tilde{W}_{22} & \tilde{W}_{23} \\ \tilde{W}_{32} & \tilde{W}_{33} \\ \tilde{W}_{42} & \tilde{W}_{43} \end{bmatrix} \cdot \begin{bmatrix} \breve{A}_{21} & \breve{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix}.
\end{aligned}$$

By our choice of T_x, the matrix before the first equal sign has full column rank and spans the subspace in (3.9). Thus, $\widetilde{X}$ is a stabilizing solution of the projected Lur'e equations if and only if $-\lambda E_{33} + A_{33}$ has full row rank for all $\lambda \in \mathbb{C}^+$. ∎

THEOREM 4.2. *Under the assumptions and notation of Theorem* 4.1*, the followiung statements hold true for the pencil* $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ *in* (3.10)*:*

(a) *If, for* $\lambda \in \mathbb{C}^-$*, there holds* $\mathcal{W}_\lambda \subset \breve{\mathcal{V}}$*, then the EKCF of* $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ *does not have blocks of type* E1 *corresponding to the generalized eigenvalue* λ*. Moreover, all blocks of type* E4 *in the EKCF of* $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ *are of size* 1×1*.*

(b) *If, for* $\lambda \in i\mathbb{R}$*, there holds* $\mathcal{V}_\lambda \subset \breve{\mathcal{V}}$*, then the EKCF of* $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ *does not have blocks of type* E2 *corresponding the the generalized eigenvalue* λ*. Moreover, all blocks of type* E4 *in the EKCF of* $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ *are of size* 1×1*.*

(c) *If there holds* $\mathcal{V}_\infty \subset \breve{\mathcal{V}}$*, then all blocks of type* E3 *and* E4 *in the EKCF of* $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ *are of size* 1×1*.*
Moreover, $\widetilde{X}$ *is the stabilizing solution of the projected Lur'e equations* (1.11) *if and only if* $\widetilde{X}$ *fulfills the projected algebraic Riccati equation*

$$\widetilde{A}^*\widetilde{X} + \widetilde{X}\widetilde{A} - (\widetilde{X}\widetilde{B} + \widetilde{S})\widetilde{R}^+(\widetilde{X}\widetilde{B} + \widetilde{S})^* + \widetilde{Q} = 0, \quad \widetilde{X} = \Pi^*\widetilde{X}\Pi \qquad (4.8a)$$

with the additional property that all generalized eigenvalues of the pencil

$$-s\Pi + \widetilde{A} - \widetilde{B}\widetilde{R}^+(\widetilde{X}\widetilde{B} + \widetilde{S})^* \qquad (4.8b)$$

are in $i\mathbb{R} \cup \mathbb{C}^-$*.*

Proof. Solvability of the projected Lur'e equations implies that, in the EKCF of $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$, all blocks of type E2 have even size and all blocks of type E3 have odd size. Assume that $\widetilde{V} \in \mathbb{C}^{2n+m,n-\breve{n}+m}$, $\widetilde{W} \in \mathbb{C}^{2n+m,n-\breve{n}+p}$, $s\hat{E} - \hat{A} \in \mathbb{C}[s]^{n-\breve{n}+m,n-\breve{n}+p}$ with $\operatorname{im}\widetilde{V} = \operatorname{im}\widetilde{X} \times \operatorname{im}\Pi \times \mathbb{C}^{m+\breve{n}}$ and

$$(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})\widetilde{V} = \widetilde{W}(s\hat{E} - \hat{A}).$$

Then the following connection between the EKCF of $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ and the KCF of $s\hat{E} - \hat{A}$ holds true:

(i) The EKCF of $s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}$ has a block of type E1 with size $2k_j\times 2k_j$ corresponding to the generalized eigenvalues $\lambda, -\overline{\lambda}$ with $\lambda\in\mathbb{C}^-$ if and only if the KCF of $s\hat{E}-\hat{A}$ a block of type K1 with size k_j corresponding to the generalized eigenvalue λ.

(ii) The EKCF of $s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}$ has a block of type E2 with size $k_j\times k_j$ corresponding to the generalized eigenvalue $\lambda\in i\mathbb{R}$ if and only if the KCF of $s\hat{E}-\hat{A}$ a block of type K1 with size $k_j/2\times k_j/2$ corresponding to the generalized eigenvalue λ.

(iii) The EKCF of $s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}$ has a block of type E3 with size $k_j\times k_j$ if and only if the KCF of $s\hat{E}-\hat{A}$ a block of type K2 with size $(k_j+1)/2\times(k_j+1)/2$.

(iv) The EKCF of $s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}$ has a block of type E4 with size $(2k_j-1)\times(2k_j-1)$ if and only if the KCF of $s\hat{E}-\hat{A}$ a block of type K3 with size $(k_j-1)\times k_j$.

By (4.7), we may assume that $s\hat{E}-\hat{A}$ is of the form

$$s\hat{E}-\hat{A}=\begin{bmatrix}\breve{A}_{21} & \breve{A}_{22} & -sE_{23}+A_{23}\\ 0 & 0 & -sE_{33}+A_{33}\end{bmatrix}.$$

Now using Lemma 2.9 we obtain the following facts:

(i') If, for $\lambda\in\mathbb{C}^-$, there holds $\mathcal{W}_\lambda\subset\breve{\mathcal{V}}$, then $\lambda E_{33}-A_{33}$ has full row rank.

(ii') If, for $\lambda\in i\mathbb{R}$, there holds $\mathcal{V}_\lambda\subset\breve{\mathcal{V}}$, then $\lambda E_{33}-A_{33}$ has full row rank.

(iii') If $\mathcal{V}_\infty\subset\breve{\mathcal{V}}$, then E_{33} has full row rank.

Statements (a) and (b) of Theorem 4.2 are then immediate consequences of (i), (ii), (iv), (i'), and (ii'). It remains to show (c): If $\mathcal{V}_\infty\subset\breve{\mathcal{V}}$, then, by (iii'), we have that E_{33} has full row rank. Assuming that the EKCF of $s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}$ has a block of type E3 which is of size greater than 1×1, we obtain by (ii) that the KCF of

$$\begin{bmatrix}\breve{A}_{21} & \breve{A}_{22} & -sE_{23}+A_{23}\\ 0 & 0 & -sE_{33}+A_{33}\end{bmatrix}$$

has a block of type K2 with size greater than 1×1. Then there exist nonzero vectors

$$z_0=\begin{bmatrix}z_{01}\\ z_{02}\end{bmatrix},\quad z_1=\begin{bmatrix}z_{11}\\ z_{12}\end{bmatrix}$$

with

$$\begin{bmatrix}z_{01}\\ z_{02}\end{bmatrix}^*\begin{bmatrix}0 & 0 & E_{23}\\ 0 & 0 & E_{33}\end{bmatrix}=0,$$
$$\begin{bmatrix}z_{01}\\ z_{02}\end{bmatrix}^*\begin{bmatrix}\breve{A}_{21} & \breve{A}_{22} & A_{23}\\ 0 & 0 & A_{33}\end{bmatrix}=\begin{bmatrix}z_{11}\\ z_{12}\end{bmatrix}^*\begin{bmatrix}0 & 0 & E_{23}\\ 0 & 0 & E_{33}\end{bmatrix}.$$

Since $\begin{bmatrix}\breve{A}_{21} & \breve{A}_{22}\end{bmatrix}$ has full row rank, the latter equation gives rise to $z_{01}=0$. The first equation together with E_{33} having full row rank then implies $z_{02}=0$. This is a contradiction.

To complete the result, we have to show that the projected Lur'e equations can be transformed into a projected Riccati equation, if $\mathcal{V}_\infty\subset\breve{\mathcal{V}}$: Since the blocks of type E3 and E4 in the EKCF of $s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}$ have size at most 1, the Wong sequence $\mathcal{V}_\infty^{(\ell)}=\mathcal{W}_\infty^{(\ell)}$ stagnates at $\ell=1$. In particular, $\ker\widetilde{L}=\ker\widetilde{R}\subset\ker\widetilde{B}\cap\ker\widetilde{S}$, as otherwise the sequence would continue. This implies

$$\widetilde{K}^*J\widetilde{K}=(\widetilde{B}\widetilde{X}+\widetilde{S})^*\widetilde{R}^+(\widetilde{B}\widetilde{X}+\widetilde{S}).$$

Plugging this into $\widetilde{A}^*\widetilde{X} + \widetilde{X}\widetilde{A} + \widetilde{Q} = \widetilde{K}^*J\widetilde{K}$, we obtain that $\widetilde{X}$ solves the projected Riccati equation (4.8a). Furthermore, since

$$\begin{bmatrix} -s\Pi + \widetilde{A} & \widetilde{B} \\ \widetilde{K} & \widetilde{L} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\widetilde{L}^-\widetilde{K} & I \end{bmatrix} = \begin{bmatrix} -s\Pi + \widetilde{A} - \widetilde{B}\widetilde{R}^-(\widetilde{X}\widetilde{B} + \widetilde{S})^* & \widetilde{B} \\ 0 & \widetilde{L} \end{bmatrix},$$

the finite generalized eigenvalues of $-s\Pi + \widetilde{A} - \widetilde{B}\widetilde{R}^-(\widetilde{X}\widetilde{B} + \widetilde{S})^*$ equal to those of $sE_X - A_X$, i.e., they are contained in $i\mathbb{R} \cup \mathbb{C}^-$. $\square$

Remark 4.3. In many cases of practical relevance, such as in the *positive real lemma* [2], the *bounded real lemma* [1], in H_∞ control [21] or the case of positive semidefinite cost functional [50], that is

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0,$$

there is an a priori knowledge of the stabilizing solution X being semidefinite. Then, we can choose $\check{V}_x^-$ in a special way that simplifies the expressions (4.5). Consider the matrix $G := \check{V}_x^* X \check{V}_x = \check{V}_\mu^* \check{V}_x = \check{V}_x^* \check{V}_\mu$. Since $X \geq (\leq) 0$, we have $G \geq (\leq) 0$ and $\ker \check{V}_\mu \subset \ker G$, and thus we can write $\check{V}_\mu = \check{W}G$ for some $\check{W} \in \mathbb{C}^{n,\check{n}}$. Then, for a left inverse $\check{V}_x^-$ of $\check{V}_x$, we can verify that

$$\check{V}_x^{=} = (I_{\check{n}} - G^+G)\check{V}_x^- + G^+\check{V}_\mu^*$$

is another left inverse of $\check{V}_x$ and satisfies

$$\check{V}_\mu^*(I_n - \check{V}_x\check{V}_x^{=}) = \check{V}_\mu^* - (G - GG^+G)\check{V}_x^- - GG^+G\check{W}^* = \check{V}_\mu - G\check{W}^* = 0.$$

Therefore, if we use $\check{V}_x^{=}$ instead of $\check{V}_x^-$ in our computations, then $\check{V}_\mu^*\Pi = 0$ holds. With this additional property, the matrices in (4.5) simplify to

$$\begin{aligned} &\widetilde{A} = \Pi A \Pi, \quad \widetilde{Q} = \Pi^* Q \Pi, \quad \widetilde{B} = \begin{bmatrix} \Pi A \check{V}_x & \Pi B \end{bmatrix}, \\ &\widetilde{S} = \begin{bmatrix} \Pi^* A^* \check{V}_\mu + \Pi^* Q \check{V}_x & \Pi^* S \end{bmatrix}, \\ &\widetilde{R} = \begin{bmatrix} \check{V}_x^* A^* \check{V}_\mu + \check{V}_\mu^* A \check{V}_x + \check{V}_x^* Q \check{V}_x & \check{V}_\mu^* B + \check{V}_x^* S \\ B^* \check{V}_\mu + S^* \check{V}_x & R \end{bmatrix}, \end{aligned} \tag{4.9}$$

and, by (4.2), the stabilizing solution is given by

$$X = \widetilde{X} + \check{V}_\mu G^+ \check{V}_\mu^*, \tag{4.10}$$

where $\widetilde{X}$ is the stabilizing solution of the projected Lur'e equations (1.11) with matrices as in (4.9). In particular, given a solution $\widetilde{X} = \pm\widetilde{Z}\widetilde{Z}^*$ in factored form, we obtain a factorization $X = \pm ZZ^*$, where $Z = [\widetilde{Z} \quad \check{V}_\mu \check{Y}]$ and $\check{Y}$ is some matrix with $\pm\check{Y}\check{Y}^* = G$. Solutions in this factored form are essential in balancing-related model order reduction and are provided by several algorithms for the solution of algebraic Riccati equations [8, 6].

5. Numerical aspects. The results in Theorems 4.1 and 4.2 can be used to develop an algorithm for the computation of the stabilizing solutions of Lur'e equations. The raw procedure can be outlined as follows.

(1) For $\ell = 1, 2, \ldots$, determine matrices $V_\infty^{(\ell)}$ with full column rank and $\mathcal{V}_\infty^{(\ell)} = \operatorname{im} V_\infty^{(\ell)}$, until they stagnate to $\mathcal{V}_\infty = \operatorname{im} V_\infty$.
(2) Solve the projected Riccati equation (4.8a) for $\widetilde{X}$.
(3) Add a low-rank term to $\widetilde{X}$, according to (1.10) or (4.10), to recover the solution of the Lur'e equation.

As Riccati solvers in step 2, we can use algorithms suited for large-scale problems [6, 41], which typically return solutions in low-rank factored form. As we see in the following, it is possible to adapt these algorithms to work in our setting of projected Riccati equations preserving the necessary sparsity. The first two steps are described in more detail in the next subsections.

Remark 5.1.

(a) In practically relevant examples, we often have $m \ll n$ and further, so the Wong sequence corresponding to the infinite eigenvalue usually stagnates after only a couple of steps. Therefore, step (1) is extremely fast, and the bulk of the computation is in step (2).
(b) The first step involves many successive nullspace determinations and may therefore be arbitrarily ill-conditioned. However, these kernels can often be obtained from considerations on structural properties of the system, e.g., in the equations of the generalized positive real lemma for equations of linear electrical circuits [38]. To exemplify this statement, we will consider a practically relevant class of problems in Example 5.4.
(c) The first step could, by Theorems 4.1 and 4.2, be extended by computing the Wong sequences corresponding to eigenvalues with negative real part, or neutral Wong sequences corresponding to purely imaginary generalized eigenvalues.
(d) The approach of deflating "critical eigenvalues" can be also applied to algebraic Riccati equations. If some critical generalized eigenvalues of the even matrix pencil (or, equivalently, some critical eigenvalues of the Hamiltonian matrix [32]) are known a priori, these can be deflated to obtain a projected algebraic Riccati equation with nicer structural properties.

5.1. Computation of Wong sequences. While basis matrices of the spaces of Wong sequences are in principle explicitly computable from (2.5) and (2.8), some care is required in the implementation, especially in the case of a large-scale problem.

An essential step in the computation of the Wong sequence corresponding to a generalized eigenvalue $\lambda \in \mathbb{C}$ (note that infinite eigenvalues can be treated analogously) is the determination of the preimage

$$\mathcal{W}_\lambda^{(\ell)} = (\lambda\mathcal{E} - \mathcal{A})^{-1}(\mathcal{E}\mathcal{W}_\lambda^{(\ell-1)}).$$

This can be done as follows. Consider basis matrices T, U, V of $\ker(\lambda\mathcal{E} - \mathcal{A})^*$, $\ker(\lambda\mathcal{E} - \mathcal{A})$, and $\mathcal{E}\mathcal{W}_\lambda^{(\ell-1)}$, respectively, and S a basis matrix of $\ker T^*V$. Notice that the equation $(\lambda\mathcal{E} - \mathcal{A})x = b$ is solvable if and only if $T^*b = 0$, thus $\operatorname{im} VS = \operatorname{im} V \cap \operatorname{im}(\lambda\mathcal{E} - \mathcal{A})$. In particular, the equation $(\lambda\mathcal{E} - \mathcal{A})X = VS$ is solvable and for any solution X, there holds $\mathcal{W}_\lambda^{(\ell)} = \operatorname{im} X + \operatorname{im} U$. This computation is feasible whenever T and U are stably computable or explicitly available due to structural properties of the involved matrices.

Note that, if $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ is a regular pencil, then $\operatorname{im} X_k \cap \operatorname{im} U = \{0\}$. In this case, line 14 in Algorithm 1 can be replaced with $W_k := \begin{bmatrix} X_k & U \end{bmatrix}$. Furthermore, since

ALGORITHM 1. Computation of $\mathcal{W}_\lambda$.

Data: a matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$, $\lambda \in \mathbb{C} \cup \{\infty\}$
Result: a basis matrix W_k of $\mathcal{W}_\lambda$

```
1  if λ = ∞ then
2  |  M := E, N := A;
3  else
4  |  M := λE − A, N := E;
5  end
6  Compute a basis matrix T of ker M*;
7  Compute a basis matrix U of ker M;
8  W_1 := U, k := 1;
9  repeat
10 |  k := k + 1;
11 |  Z_k := N Z_{k−1};
12 |  Compute a basis matrix S of ker T* W_k;
13 |  Solve M X_k = Z_k S_k for the matrix X_k;
14 |  Compute a basis matrix W_k of im [X_k  U];
15 until rank W_k = rank W_{k−1};
```

W_k, W_{k-1} have full column rank, the stop criterion $\operatorname{rank} W_k = \operatorname{rank} W_{k-1}$ reduces to a simple comparison of the numbers of rows of W_k and W_{k-1}.

For $\lambda \in i\mathbb{R} \cup \{\infty\}$ and an even matrix pencil $s\mathcal{E} - \mathcal{A}$, the computation of a basis matrix of $\mathcal{V}_\lambda$ additionally involves the step $\mathcal{Z}_\lambda^{(\ell)} \cap (\mathcal{Z}_\lambda^{(\ell)})^{\mathcal{E}\perp}$. Computation of the latter subspace is based on the following result.

LEMMA 5.2. *For skew-symmetric $\mathcal{E} \in \mathbb{C}^{N,N}$ and $U \in \mathbb{C}^{N,M}$, there holds*

$$\operatorname{im} U \cap (\operatorname{im} U)^{\mathcal{M}\perp} = U \cdot \ker(U^* \mathcal{E} U).$$

Proof. For $w \in \ker(U^*\mathcal{E}U)$, simple arithmetic leads to $Uw \in \operatorname{im} U \cap (\operatorname{im} U)^{\mathcal{E}\perp}$. Hence, $\operatorname{im} U \cap (\operatorname{im} U)^{\mathcal{E}\perp} \supset U \cdot \ker(U^*\mathcal{E}U)$.

For showing the converse inclusion, let $u = Uw \in \operatorname{im} U \cap (\operatorname{im} U)^{\mathcal{E}\perp}$. Then, by $Uw \in (\operatorname{im} U)^{\mathcal{E}\perp}$, there holds $(Uw)^*\mathcal{E}U = 0$, whence $w \in \ker(U^*\mathcal{E}U)$. This gives rise to

$$u = Uw \in U \cdot \ker(U^*\mathcal{E}U). \qquad \square$$

Using this result, we can extend Algorithm 1 to determine the $\mathcal{E}$-neutral deflating subspace corresponding to a generalized eigenvalue $\lambda \in i\mathbb{R} \cup \{\infty\}$. Note that, for $\lambda \in i\mathbb{R}$, the matrix $\lambda\mathcal{E} - \mathcal{A}$ is Hermitian. Since, moreover, $\mathcal{E}$ is skew-Hermitian, we may choose $T = U$ in the notation of Algorithm 1.

Remark 5.3.

(a) Some further extensions are possible to further improve numerical efficiency in the computation of $\mathcal{W}_\lambda$ and $\mathcal{V}_\lambda$. For instance, we may consider at every step only a basis of a space $\mathcal{P}^{(\ell)}$ such that $\mathcal{V}_\lambda^{(\ell-1)} \oplus \mathcal{P}^{(\ell)} = \mathcal{V}_\lambda^{(\ell)}$. However, in the case where $\dim \mathcal{V}_\lambda$ is small, this improvement is only very little.

(b) In the computation of $\mathcal{V}_\infty$ for the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.9), the structure of the pencil can be exploited in several parts of the computation. Namely, we know in advance that $U = \begin{bmatrix} 0_{m,n} & 0_{m,n} & I_m \end{bmatrix}^T$, $V_1 = R$, multiplication by $\mathcal{N} = \mathcal{A}$ can be performed exploiting the sparsity and/or low rank properties of A and Q, and $X_k = \mathcal{E}^+(V_k S_k) = -\mathcal{E} V_k S_k$. Then the computation in line 13 in Algorithm 1 reduces to $Z_k = [\, X_k \,, \, U \,]$.

ALGORITHM 2. Computation of $\mathcal{V}_\lambda$.

Data: an even matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$, $\lambda \in i\mathbb{R} \cup \{\infty\}$
Result: a basis matrix V_k of $\mathcal{V}_\lambda$

if $\lambda = \infty$ **then**
  $\mathcal{M} := \mathcal{E}$, $\mathcal{N} := \mathcal{A}$;
else
  $\mathcal{M} := \lambda\mathcal{E} - \mathcal{A}$, $\mathcal{N} := \mathcal{E}$;
end
Compute a basis matrix U of $\ker \mathcal{M}$;
$Z_1 := U$, $V_1 := U$, $k := 1$;
repeat
  $k := k + 1$;
  $V_k := \mathcal{N} V_{k-1}$;
  Compute a basis matrix S_k of $\ker U^* V_k$;
  Solve $\mathcal{M} X_k = V_k S_k$ for the matrix X_k;
  Compute a basis matrix Z_k of $\operatorname{im} \begin{bmatrix} X_k & U \end{bmatrix}$;
  Compute a basis matrix Y_k of $\ker Z_k^* \mathcal{E} Z_k$;
  Compute a basis matrix V_k of $\operatorname{im} \begin{bmatrix} V_k & Z_k Y_k \end{bmatrix}$;
until $\operatorname{rank} V_k = \operatorname{rank} V_{k-1}$;

An important issue to discuss is the numerical stability of the rank determinations. In Algorithms 1 and 2, all rank determinations take the form of computing basis matrices for images or kernels of some explicitly computed matrix M; this can be done using the SVD; all rank decisions correspond now to choosing a threshold under which the computed singular values are considered to be zero. In a similar fashion to the implementation of MATLAB's `orth` and `null` functions, in our we used computations the convention that a singular value σ_k of a $p \times q$ matrix is considered to be zero if

$$\sigma_k \leq \max(p, q)\sigma_1 \epsilon, \tag{5.1}$$

where we chose $\epsilon = \mathbf{u}^{1/2}$, with $\mathbf{u} \approx 2.2 \cdot 10^{-16}$ the machine precision. However, it is simple to keep track of the sensitivity of this decision; in our experiments, we checked which values $\epsilon \geq \mathbf{u}$ would yield the same result if plugged in the previous expression. A narrow interval means that the rank decisions are ill-posed.

The following example underlines that structural properties of the system may be employed to obtain a basis matrix of $\mathcal{V}_\infty$ without numerically invoking Algorithm 2.

Example 5.4 ((strictly) positive real systems). As mentioned in the introduction, one of the important applications of Lur'e equations is in the *positive real lemma* [2]: Given are matrices $A \in \mathbb{R}^{n,n}$ and $B, C^T \in \mathbb{R}^{n,m}$ with the property that (A, B) is controllable and $G(s) = C(sI - A)^{-1}B \in \mathbb{R}(s)^{m,m}$ fulfills $G(\lambda) + G(\lambda)^* \geq 0$ for all $\lambda \in \mathbb{C}^+$, the Lur'e equations (1.1) with $R = 0$, $Q = 0$, $S = -C^T$, and $J = -I_m$ are known to admit a positive definite stabilizing solution X. Indeed, $\Phi(s) = -G(s) - G(-s)^T$ is a spectral density function corresponding to the Lur'e equations from the positive real lemma. A subclass of particular interest are the so-called *strictly positive real systems* [49]. That is, there exists some $\varepsilon > 0$ such that $G(\lambda) + G(\lambda)^* > 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > -\varepsilon$. It is shown in [49] that this is equivalent to

(i) $G(i\omega) + G(i\omega)^* > 0$ for all $\omega \in \mathbb{R}$ and

(ii) $\lim_{\omega\to\infty} \omega^2(G(i\omega) + G(i\omega)^*) > 0$.

Note that, by using a special congruence transformation of $i\omega\mathcal{E} - \mathcal{A}$ (see (3.1) in [40]), it can be shown that property (i) implies that the associated even matrix pencil does not have any generalized eigenvalues on the imaginary axis. Moreover, by forming the Laurent expansion $G(s) = \sum_{k=1}^{\infty} s^{-k} CA^k B$, it follows that property (ii) is equivalent to $CB = B^T C^T > 0$ and $CAB + B^T A^T C^T > 0$.

Now we perform Algorithm 2 to determine the space $\mathcal{V}_\infty$: Since $\mathcal{V}_\infty^{(1)} = \ker \mathcal{E}$, we can choose $U = V_1 = \begin{bmatrix} 0 & 0 & I_m \end{bmatrix}$. Using that CB is symmetric, we obtain in the second step that

$$Z_2 = V_2 = \begin{bmatrix} -C^T & 0 \\ -B & 0 \\ 0 & I_m \end{bmatrix}.$$

A basis matrix of $\mathcal{E}^{-1}(\mathcal{A} \cdot \operatorname{im} V_2)$ is then given by

$$Z_3 = \begin{bmatrix} A^T C^T & C^T & 0 \\ -AB & B & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

To determine a basis matrix V_3, we need to consider the kernel of the matrix

$$Z_3^* \mathcal{E} Z_3 = \begin{bmatrix} CA^2B - B^T(A^T)^2C^T & -CAB - B^TA^TC^T & 0 \\ CAB + B^TA^TC^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using that $CAB + B^T A^T C^T > 0$, we have $\ker Z_3^* \mathcal{E} Z_3 = \{0\} \times \{0\} \times \mathbb{R}^m$. As a consequence, we have $V_3 = V_2$, i.e., the algorithm stagnates here.

According to Theorem 4.1, the stabilizing solution fulfills $XB = C$. In the notation of Theorem 4.1, we have $\check{V}_x = B$ and $\check{V}_\mu = C^T$. To determine the projected algebraic Riccati equation describing the "remaining part," we make use of $CB > 0$ to construct a left inverse $\check{V}_x^- = (CB)^{-1}C$ of $\check{V}_x$. The projector (4.1) is consequently given by $\Pi = I - B(CB)^{-1}C$; the remaining projected Riccati equation reads, in compact form,

$$\begin{aligned} 0 &= (\Pi A \Pi)^T \widetilde{X} + \widetilde{X}(\Pi A \Pi) \\ &\quad + (\widetilde{X}\Pi AB - \Pi^T A^T C^T)(CAB + B^T A^T C^T)^{-1}(\widetilde{X}\Pi AB - \Pi^T A^T C^T)^T. \end{aligned}$$

5.2. Numerical solution of projected Riccati equations. We consider projected Riccati equations

$$A_R^* \widetilde{X} + \widetilde{X} A_R + H_R - \widetilde{X} G_R \widetilde{X} = 0, \quad \widetilde{X} = \Pi^* \widetilde{X} \Pi \tag{5.2a}$$

with

$$A_R = \Pi A_R \Pi, \quad H_R = \Pi^* H_R \Pi, \quad G_R = \Pi G_R \Pi^*. \tag{5.2b}$$

These are obtained from Lur'e equations after the deflation process described in Theorems 4.1 and 4.2 (see section 4 for computational issues) with matrices

$$A_R = \widetilde{A} - \widetilde{B}\widetilde{R}^+\widetilde{S}^* \quad H_R = \widetilde{Q} - \widetilde{S}\widetilde{R}^+\widetilde{S}^*, \quad G_R = \widetilde{B}\widetilde{R}^+\widetilde{B}^*. \tag{5.3}$$

In theory, a change of coordinates with T as in (3.7) transforms this equation into a conventional algebraic Riccati equation bordered by zero blocks:

$$\begin{bmatrix} A_{R11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{R11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H_{R11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{R11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.4}$$

However, we aim to solve the projected algebraic Riccati equation (5.2) without actually changing coordinates as above, in order to preserve sparsity. Different numerical methods exist for the solution of large-scale AREs; for instance, the Newton–Galerkin method in [9] or the Newton–Kleinman method [31] coupled with an effective method for solving the resulting Lyapunov equations, such as the ADI iteration [6], the K-PIK method [41], or in the symmetric case direct Riemannian optimization methods [45]. If the quadratic term is indefinite (which corresponds to indefinite J), the method of Lanzon et al. [33] may be applied.

Note that the Newton–Kleinman method is quadratically convergent if the pencil

$$-s\Pi + \widetilde{A} - \widetilde{B}\widetilde{R}^{+}(\widetilde{X}\widetilde{B} + \widetilde{S})^*$$

does not have purely imaginary generalized eigenvalues or, equivalently, the EKCF of $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ does not have any blocks of type E2. By Theorem 4.2, the latter property is fulfilled if $s\mathcal{E} - \mathcal{A}$ does not have any imaginary generalized eigenvalues or if

$$\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \subset \breve{\mathcal{V}}.$$

Moreover, all these methods apart from the last are essentially based on rational Krylov subspaces and the solution of a large number of linear systems of the form

$$(A_{R11} + pI)x_1 = b_1 \quad \text{or} \quad (A_{R11}^* + pI)x_1 = b_1, \tag{5.5}$$

for suitable vectors $x_1, b_1 \in \mathbb{C}^{n-\breve{n}}$.

In the case of our projected Riccati equation, the key feature that allows us to preserve sparsity is that we can recover the solution of these linear systems through computations involving A only, without having to apply explicitly the change of basis T or the projector Π, which would destroy sparsity. We focus on the first of the two forms, as the other case is essentially the same. Indeed, if x_1 solves the first system in (5.5), then

$$\begin{bmatrix} A_{R11} + pI & 0 \\ 0 & pI \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}.$$

So instead of x_1 and b_1 we can get implicitly the solution of the same linear system by solving

$$(A_R + pI_n)^{-1}x = b, \quad x := T \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad b := T \begin{bmatrix} b_1 \\ 0 \end{bmatrix}.$$

The aforementioned algorithms for the Riccati equation (5.4) can then be rewritten by working only with vectors in the form

$$v = T \begin{bmatrix} v_1 \\ 0 \end{bmatrix}; \tag{5.6}$$

this transformation indeed preserves both rational Krylov subspaces built upon A and linear system solutions. Numerical drift in the zero in the second component can be corrected by reapplying the projector Π, and the initial values needed for the

Krylov subspaces are typically obtained from the low-rank factors of H_{R11} and are thus naturally obtained in the form (5.6) when we compute the low-rank factors of H_R. In no place in these algorithms need we apply explicitly the change of basis T.

If one wishes to have the linear system matrix explicitly represented as a sparse matrix, rather than as the expression $A_R = \Pi A \Pi - \widetilde{B}\widetilde{R}^+\widetilde{S}^*$ with $\Pi = I_n - \check{V}_x \check{V}_x^-$, this is possible by considering the extended system

$$\begin{bmatrix} A + pI & \widetilde{B} & \check{V}_x & A\check{V}_x \\ \widetilde{R}^+\widetilde{S}^* & I_p & 0 & 0 \\ \check{V}_x^- A - \check{V}_x^- A \check{V}_x \check{V}_x^- & 0 & I_{\check{n}} & 0 \\ \check{V}_x^- & 0 & 0 & I_{\check{n}} \end{bmatrix} \begin{bmatrix} x \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.7}$$

We can see that x in (5.7) solves $(A_R + pI)^* x = b$. If the dimension of the space $\check{\mathcal{V}}$ as in (3.6) is moderate and A is sparse, then the extended system matrix can be stored in sparse form. Therefore by writing the system in this form we can use a direct sparse solver (such as sparse LU) or general preconditioners (such as incomplete LU).

6. Numerical examples. As a numerical experiment, we consider Lur'e equations arising in the *positive real lemma* [2] (see also Example 5.4): Given are matrices $A \in \mathbb{R}^{n,n}$ and $B, C^T \in \mathbb{R}^{n,m}$ with the property that (A, B) is controllable and $G(s) = C(sI - A)^{-1}B \in \mathbb{R}(s)^{m,m}$ fulfills $G(\lambda) + G^*(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}^+$, the Lur'e equations (1.1) with $R = 0$, $Q = 0$ and $S = -C^T$ and $J = -I_m$ are known to admit a positive definite stabilizing solution $X \in \mathbb{R}^{n,n}$.

In the considered examples, we have $A + A^* \leq 0$ and $B = C^T$, which implies positive realness of $G(s)$. We have taken dynamical systems in the benchmarks examples `demo_m1`, `demo_r1`, and `demo_r3` from the MATLAB library LYAPACK library [35].

For the corresponding Lur'e equations, we compute the subspace $\check{\mathcal{V}} = \mathcal{V}_\infty$ by Algorithm 2. The generalized inverse $\check{V}_x^-$ has been chosen in a way that $\check{V}_\mu^* \Pi = 0$ (see Remark 4.3). The obtained projected algebraic Riccati equation (5.2) is solved with the Newton–Kleinman-ADI method using the LYAPACK.

We use the library by providing a custom solver for both shifted and unshifted linear equations, according to the remarks in section 5.2. In particular, we rely on the library's heuristic for the choice of the shift parameters. In the considered examples, the Newton–Kleinman iteration may be stably initialized with $X^{(0)} = 0$. After obtaining the solution $\widetilde{X}$ of the projected Riccati equation, we recover the Lur'e solution as $X = \widetilde{X} + \check{V}_\mu \check{V}_x^-$.

Computations were done on Intel Core 2 Duo CPU E6750 @2.66GHz with machine precision $\mathbf{u} = 2.22 \times 10^{-16}$ using MATLAB R2010b. We report the results of the experiments in Table 6.1. The relative residual of the Lur'e equations is measured as

$$\text{Res} = \frac{\left\| \mathfrak{L}(X) - \begin{bmatrix} K^* \\ L^* \end{bmatrix} J \begin{bmatrix} K & L \end{bmatrix} \right\|_F}{\|\mathfrak{L}(X)\|_F}, \quad \mathfrak{L}(X) = \begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix}, \tag{6.1}$$

where the missing solution components K and L are computed by truncating to rank m an eigendecomposition of $\mathfrak{L}(X)$. Notice that in most application only X is needed, so we need only this expensive computation if we want to check the residual.

To check whether the computed solution is the stabilizing one, we construct the reduced pencil associated to this deflating subspace

$$s\widehat{\mathcal{E}} - \widehat{\mathcal{A}} = \begin{bmatrix} -sI + A & B \\ K & L \end{bmatrix}$$

TABLE 6.1
Results of the numerical experiments.

	demo_m1	demo_r1	demo_r3
n	408	2500	821
m	1	1	6
$\check{n}$	1	1	6
Rtol (see (6.3))	$(2.22 \times 10^{-16}$, $1.412{,}7 \times 10^{-03})$	$(2.22 \times 10^{-16}$, $1.563{,}3 \times 10^{-04})$	$(2.22 \times 10^{-16}$, $1.092{,}7 \times 10^{-06})$
ADI itns for computing $X^{(1)}$	41	39	44
rank of $X^{(1)}$	25	24	138
rank of $X - X^{(1)}$	28	23	130
number of Newton steps	8	4	7
avg. ADI itns / Newton step	32.25	37.25	36.857
Res (see (6.1))	$2.430{,}6 \times 10^{-07}$	$2.591{,}8 \times 10^{-15}$	$3.457{,}9 \times 10^{-15}$
Stab (see (6.2))	$-2.851{,}6 \times 10^{-09}$	$-1.776{,}4 \times 10^{-15}$	$-1.316{,}0 \times 10^{-08}$
CPU time (seconds)	$5.519{,}1 \times 10^{+00}$	$1.712{,}2 \times 10^{+01}$	$6.466{,}0 \times 10^{+01}$

according to (3.1), and we check whether its Cayley transform $s(\widehat{\mathcal{A}}+\widehat{\mathcal{E}})-(\widehat{\mathcal{A}}-\widehat{\mathcal{E}})$ has only eigenvalues larger than 1, since this Cayley transform maps the left half-plane onto the exterior of the unit disc. We report on our table the value of

$$\text{Stab} = \min_{\lambda\in\sigma(s(\widehat{\mathcal{A}}+\widehat{\mathcal{E}})-(\widehat{\mathcal{A}}-\widehat{\mathcal{E}}))} |\lambda| - 1; \tag{6.2}$$

we expect Stab $\geq -c \cdot 10^{-8}$ for a moderate constant $c > 0$ based on the preceding discussion. Indeed, due to $R = 0$, in all our problems the EKCF of the corresponding even matrix pencil $s\mathcal{E}-\mathcal{A}$ as in (1.9) has at least one block of type $E3$ with size greater than or equal to 3×3. This implies that the KCF of $s\widehat{\mathcal{E}} - \widehat{\mathcal{A}}$ contains a block K2 of size at least 2×2; therefore, the sensitivity of the eigenvalue 1 in the computation of Stab is $\sqrt{\mathbf{u}}$.

Moreover, following the discussion at the end of section 5.1, we reported a stability measure for the rank decisions in the Wong sequence computation. Namely, we report the interval

$$\text{Rtol} = (\epsilon_-, \epsilon_+) \tag{6.3}$$

of values ϵ that we can plug in (5.1) without modifying the rank decisions. The width of this interval indicates that the difference in norm between the computed neglected and nonneglected singular values. In all cases we have a decay in the singular values that is sharp enough to ensure that the rank determination is well conditioned.

7. Conclusion. We have considered a constructive approach to the determination of the stabilizing solution of Lur'e equations. Based on the correspondence of the solution set to $\mathcal{E}$-neutral deflating subspaces of an associated even matrix pencils $s\mathcal{E} - \mathcal{A}$, we transform the Lur'e equations to a projected algebraic Riccati equation. This equation can be solved using the standard large-scale Riccati solvers, with some minor modifications for dealing with the projected part. In particular, the sparsity properties are preserved along the algorithm. Altogether, this provides a new method for the low-rank approximative numerical solution of large-scale Lur'e equations.

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