Solvability and uniqueness criteria for generalized Sylvester-type equations*

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November 16, 2016

Abstract

We provide necessary and sufficient conditions for the generalized $\star$-Sylvester matrix equation, $AXB + CX^\star D = E$, to have exactly one solution for any right-hand side E. These conditions are given for arbitrary coefficient matrices A, B, C, D (either square or rectangular) and generalize existing results for the same equation with square coefficients. We also review the known results regarding the existence and uniqueness of solution for generalized Sylvester and $\star$-Sylvester equations.

Keywords. Sylvester equation, eigenvalues, matrix pencil, matrix equation

AMS classification: 15A22, 15A24, 65F15

1 Introduction

We consider the *generalized $\star$-Sylvester equation*

$$AXB + CX^\star D = E \qquad (1)$$

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*This work was partially supported by the Ministerio de Economía y Competitividad of Spain through grants MTM2015-68805-REDT, and MTM2015-65798-P (F. De Terán), and by an INdAM/GNCS Research Project 2016 (B. Iannazzo, F. Poloni, and L. Robol). Part of this work was done during a visit of the first author to the Università di Perugia as a Visiting Researcher.

for the unknown $X \in \mathbb{C}^{m\times n}$, with $\star$ being either the transpose ($\top$) or the conjugate transpose ($*$), and A, B, C, D, E being matrices with appropriate sizes. We are interested in the most general situation, where both the coefficients and the unknown are allowed to be rectangular. This equation is closely related to the *generalized Sylvester equation*

$$AXB - CXD = E, \tag{2}$$

and they are natural extensions of the *Sylvester equation* and the $\star$-*Sylvester equation*, $AX - XD = E$ and $AX + X^\star D = E$, respectively.

Sylvester-like equations are among the most popular matrix equations, and they arise in many applications (see, for instance, [1–3, 14] and the recent review [17]). In particular, equations with rectangular coefficients arise in several eigenvalue perturbation and updating problems [13, 18].

As with every class of equations, the two most natural questions regarding (1) and (2) are:

S (solvability). Does the equation have a solution, for given A, B, C, D, E?

US (unique solvability). Does the equation have exactly one solution, for given A, B, C, D, E?

Moreover, three additional questions arise naturally for linear equations, due to their peculiar structure:

SR (solvability for any right-hand side). Given A, B, C, D, does the equation have *at least* one solution for any choice of the right-hand side E?

OR (at most one solution for any right-hand side). Given A, B, C, D, does the equation have *at most* one solution for any choice of the right-hand side E?

UR (unique solvability for any right-hand side). Given A, B, C, D, does the equation have *exactly* one solution for any choice of the right-hand side E?

Using vectorizations, equations (1) and (2) can be transformed into a system of linear equations (either over the real or the complex field) of the form $Mx = e$, where M depends only on the coefficients A, B, C, D, and e depends only on the right-hand side E. This is clear for (2), where $M = B^\top \otimes A - D^\top \otimes C$ (see e.g. [10, Section 4.3]), and can be established with a little more effort for (1) (see Section 1.2). In this setting, question **SR** is equivalent to asking whether M has full row rank, and question **OR** is equivalent to asking whether M has full column rank. Question **OR** is also equivalent to asking whether the homogeneous equation ($E = 0$) has only the trivial solution $X = 0$. In order for the answer of question **UR** to be affirmative, M must be square and, in this case, all **US**, **SR**, and **OR** are equivalent to **UR**.

Instead of the conditions on the matrix M, in many cases it is possible to find conditions of a different kind, related to the spectral properties of smaller

	$AXB-CXD=E$		$AXB+CX^\star D=E$	
	square coefficients	general coefficients	square coefficients	general coefficients
S	[7, Th. 6.1]	[7, Th. 6.1], [11, Th. 1]	[7, Th. 6.1]	[7, Th. 6.1]
US	[4, Th. 1]	[11, Th. 1]	[6, Th. 15]	open
SR	same as **US**	Th. 9 (using [11])	same as **US**	open
OR	same as **US**	[12, Cor. 5],Th. 9 (using [11])	same as **US**	open
UR	same as **US**	Th. 9 (using [11])	same as **US**	Th. 3

Table 1: Existing solvability and uniqueness results for Equations (1) and (2) in terms of 'small-size' matrices and pencils.

matrix pencils. For instance, let us consider question **UR** in the case in which all matrices are square with the same size $m = n$: for Equation (2), it has a positive answer if and only if the $n \times n$ matrix pencils $A - \lambda C$ and $D - \lambda B$ are regular and do not have common eigenvalues [4, Th. 1]; and for Equation (1) the answer depends on spectral properties of the $2n \times 2n$ matrix pencil

$$\mathcal{Q}(\lambda) = \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \tag{3}$$

(see Theorem 2 for a precise statement, or [6, Th. 15] for more details). By contrast, M has size n^2 or $2n^2$. Even for equations with rectangular coefficients, the picture is the same: the characterizations that do not involve vectorization lead to matrix pencils with smaller size, of the same order as the coefficient matrices, while approaches based on Kronecker products lead to larger dimensions.

Several authors have given conditions of this kind: in Table 1, we show an overview of these results. In this table we consider separately the cases where the coefficient matrices A, B, C, D are square, and the more general case where they have arbitrary size (as long as the product is well-defined). This distinction fits with the historical flow of the problem, as can be seen in the table. Note that, in both (1) and (2), if the coefficient matrices A, B, C, D are square then the coefficient matrix M of the associated linear system is square as well. However, there is a difference between (1) and (2) regarding this issue: whereas in (2) it may happen that all A, B, C, D are square but A and B have different size, in (1) if all coefficient matrices are square, then they must all have the same size in order for the products to be well-defined.

The solution of Equation (2), allowing for rectangular coefficients, had been considered in [15], where an approach through the Kronecker canonical form of $A - \lambda C$ and $D - \lambda B$ was proposed. However, no explicit characterization for the uniqueness of solution was given in that reference. Also, [13, Th. 3.2] presents a computation of the solution space of (2) with $B = I$, depending on the Kronecker canonical form of $A - \lambda C$, but restricted to the case where this canonical form does not contain right singular blocks. A complete analysis of the solution space of Equation (2) is given by Košir in [11]; the same author gives an explicit answer to question **SR** in [12]. Answers to **OR** and **UR** follow from Košir's work, but they are not stated explicitly; for completeness, we formulate them in Section 4.

Instead, for the $\star$-Sylvester equation (1), to the best of our knowledge, several problems are still open; i.e., only characterizations requiring vectorization and Kronecker products are known. The main goal of this paper is to give a solution to **UR** in the most general case of rectangular coefficients: we give necessary and sufficient conditions for the unique solvability of (1) in terms of the pencil $\mathcal{Q}(\lambda)$ in (3).

Our focus on question **UR** is motivated by the fact that this is the only case where the operator $X \mapsto AXB + CX^\star D$ is invertible. Moreover, this is the only case where the solution of the equation is a well-posed problem, since the (unique) solution depends continuously on the (entries of the) coefficient matrices. This is no longer true in the remaining cases.

We emphasize that the approach followed in [6] for square coefficients cannot be applied in a straightforward manner to the rectangular case. Indeed, that approach is based on the characterization of the uniqueness of solution of the $\star$-Sylvester equation $AX + X^\star D = 0$ provided in [3, 14], which is valid only for $A, D, X \in \mathbb{C}^{n\times n}$. We follow a different approach that allows us to extend that characterization to the case of rectangular coefficients.

The most interesting feature of our characterization is the appearance of an additional invertibility constraint that is not present in the square case. Indeed, the unique solvability of (1) cannot be characterized completely in terms of the eigenvalues of $\mathcal{Q}(\lambda)$ only, as we show with a counterexample in Section 2.3.

1.1 The main result

In this section we state the main result of this paper, namely Theorem 3. The rest of the paper is devoted to prove this result, but, before its statement, we introduce some notation and tools.

A matrix pencil $\mathcal{P}(\lambda) = \lambda M + N$ is said to be singular if either $\mathcal{P}(\lambda)$ is rectangular or $q(\lambda) := \det\big(\mathcal{P}(\lambda)\big)$ is identically zero. If $\mathcal{P}(\lambda)$ is not singular, then it is said to be regular and the set of roots of $q(\lambda)$, complemented with ∞ if the degree of $q(\lambda)$ is less than n, is the spectrum of $\mathcal{P}$, denoted by $\Lambda(\mathcal{P})$. With $m_\lambda(\mathcal{P})$ we denote the algebraic multiplicity of the eigenvalue λ in $\mathcal{P}$. If M is a square matrix, by $\Lambda(M)$ and $m_\lambda(M)$ we denote, respectively, the spectrum of M and the algebraic multiplicity of λ as an eigenvalue of M.

The *reversal pencil* of the matrix pencil $\mathcal{P}(\lambda) = \lambda M + N$ is the pencil $\operatorname{rev}\mathcal{P}(\lambda) := \lambda N + M$. The pencil $\mathcal{P}(\lambda)$ has an infinite eigenvalue if and only if $\operatorname{rev}\mathcal{P}(\lambda)$ has the zero eigenvalue. The multiplicity of the infinite eigenvalue in $\mathcal{P}(\lambda)$ is the multiplicity of the zero eigenvalue in $\operatorname{rev}\mathcal{P}(\lambda)$.

Throughout the paper we denote by I the identity matrix of appropriate size, and by $M^{-\star}$ we denote the inverse of the matrix $M^\star$, for an invertible matrix M.

We also recall the following notion, which plays a central role in Theorem 3.

Definition 1. (Reciprocal free and $*$-reciprocal free set) [3, 14]. Let $\mathcal{S}$ be a subset of $\mathbb{C} \cup \{\infty\}$. We say that $\mathcal{S}$ is

(a) reciprocal free if $\lambda \neq \mu^{-1}$, for all $\lambda, \mu \in \mathcal{S}$;

(b) $*$-reciprocal free if $\lambda \neq (\overline{\mu})^{-1}$, for all $\lambda, \mu \in \mathcal{S}$.

This definition includes the values $\lambda = 0, \infty$, with the customary assumption $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

Before stating the characterization for the uniqueness of solution in the general case, we recall here the main result in [6], namely the characterization of the uniqueness of solution of (1) when all coefficients are square and with the same size.

Theorem 2. [6, Th. 15] *Let $A, B, C, D \in \mathbb{C}^{n \times n}$ and let $\mathcal{Q}(\lambda) = \left[\begin{smallmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{smallmatrix}\right]$. Then the equation $AXB + CX^\star D = E$ has a unique solution, for any right-hand side E, if and only if $\mathcal{Q}(\lambda)$ is regular and:*

- *If $\star = \top$, $\Lambda(\mathcal{Q}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{Q})$ is $*$-reciprocal free.*

If we allow for rectangular coefficient matrices, that is $A \in \mathbb{C}^{p \times m}, B \in \mathbb{C}^{n \times q}, C \in \mathbb{C}^{p \times n}, D \in \mathbb{C}^{m \times q}$, then several subtleties arise, and in the end, they will result in additional restrictions in the pencil $\mathcal{Q}(\lambda)$. More precisely, we will prove that the only case in which unique solvability arises is when $(p, q) \in \{(m, n), (n, m)\}$. In the case where $p = m$, that is, $A \in \mathbb{C}^{m \times m}, B \in \mathbb{C}^{n \times n}, C \in \mathbb{C}^{m \times n}, D \in \mathbb{C}^{m \times n}$, the spectrum of the matrix pencil (3) contains the infinite eigenvalue, with multiplicity at least $|m - n|$. Then, in this case we denote by $\widehat{\Lambda}(\mathcal{Q})$ the following set obtained from $\Lambda(\mathcal{Q})$:

$$\widehat{\Lambda}(\mathcal{Q}) := \left\{ \begin{array}{ll} \Lambda(\mathcal{Q}), & \text{if } m_\infty(\mathcal{Q}) > |m - n|, \\ \Lambda(\mathcal{Q}) \setminus \{\infty\}, & \text{if } m_\infty(\mathcal{Q}) = |m - n|. \end{array} \right.$$

If $p = n$, that is $A \in \mathbb{C}^{n \times m}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{m \times m}, D \in \mathbb{C}^{n \times n}$, the spectrum of the matrix pencil (3) contains the zero eigenvalue, with multiplicity at least $|m - n|$. Then, in this case we denote by $\widetilde{\Lambda}(\mathcal{Q})$ the following set obtained from $\Lambda(\mathcal{Q})$:

$$\widetilde{\Lambda}(\mathcal{Q}) := \left\{ \begin{array}{ll} \Lambda(\mathcal{Q}), & \text{if } m_0(\mathcal{Q}) > |m - n|, \\ \Lambda(Q) \setminus \{0\}, & \text{if } m_0(\mathcal{Q}) = |m - n|. \end{array} \right.$$

The presence of these additional zero/infinity eigenvalues of $\mathcal{Q}(\lambda)$ in (3) is due to the "rectangularity" of either the diagonal blocks C, D or the anti-diagonal blocks A and B. Following [9], based on the theory developed in [16], these extra zero/infinity eigenvalues are called *dimension induced* eigenvalues. The sets $\widehat{\Lambda}(\mathcal{Q})$ and $\widetilde{\Lambda}(\mathcal{Q})$ are referred to as the set of *core eigenvalues.*

With these considerations in mind, we can state the main result of this paper, which is an extension of Theorem 2 and which will be proved in Section 2.

Theorem 3. *Let* $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, *and* $D \in \mathbb{C}^{m\times q}$ *and set* $\mathcal{Q}(\lambda) := \left[\begin{smallmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{smallmatrix}\right]$. *The equation*

$$AXB + CX^\star D = E$$

has a unique solution, for any right-hand side E, *if and only if* $\mathcal{Q}(\lambda)$ *is regular and one of the following situations holds:*

(1) $p = m \neq n = q$, *either* $m > n$ *and* B *is invertible or* $m < n$ *and* A *is invertible, and*

  - *If* $\star = \top$, $\widehat{\Lambda}(\mathcal{Q}) \setminus \{\pm 1\}$ *is reciprocal free and* $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.
  - *If* $\star = *$, $\widehat{\Lambda}(\mathcal{Q})$ *is* $*$*-reciprocal free.*

(2) $p = n \neq m = q$, *either* $m > n$ *and* C *is invertible or* $m < n$ *and* D *is invertible, and*

  - *If* $\star = \top$, $\widetilde{\Lambda}(\mathcal{Q}) \setminus \{\pm 1\}$ *is reciprocal free and* $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.
  - *If* $\star = *$, $\widetilde{\Lambda}(\mathcal{Q})$ *is* $*$*-reciprocal free.*

(3) $p = m = n = q$, *and*

  - *If* $\star = \top$, $\Lambda(\mathcal{Q}) \setminus \{\pm 1\}$ *is reciprocal free and* $m_1(\mathcal{Q}) = m_{-1}(\mathcal{Q}) \leqslant 1$.
  - *If* $\star = *$, $\Lambda(\mathcal{Q})$ *is* $*$*-reciprocal free.*

1.2 Vectorization

Equation (1) can be considered as a linear system in the entries of the unknown matrix X. The natural approach to get such system is applying the vectorization (vec) operator [10, §4.3].

Set $X \in \mathbb{C}^{m\times n}$, and let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}, E \in \mathbb{C}^{p\times q}$. In the case $\star = \top$, after applying the vec operator we obtain a linear equation $M \operatorname{vec}(X) = \operatorname{vec}(E)$, with $M \in \mathbb{C}^{(pq)\times(mn)}$ given by

$$M = B^\top \otimes A + (D^\top \otimes C)\Pi, \tag{4}$$

where Π is a permutation matrix associated with the transposition [10, Equation 4.3.9b].

In the case $\star = *$, some more care is needed, since the system obtained by vectorization is not linear over $\mathbb{C}$, due to the presence of conjugations. Nevertheless, we can separate real and imaginary parts as in [6, §1.1] and write it as a linear system (over $\mathbb{R}$) of size $(2pq) \times (2mn)$ in $Y = \operatorname{vec}(\left[\begin{array}{cc} \operatorname{re}(X) & \operatorname{im}(X) \end{array}\right])$.

The fact that Equation (1) is equivalent to a linear system has two important consequences. The first one is that (1) can have a unique solution, for any right-hand-side, only if the coefficient matrix of the linear system is square, that is, $mn = pq$. The second one is that, provided that $pq = mn$, the uniqueness of solution does not depend on the right-hand side: Equation (1) has a unique solution for any E if and only if the corresponding homogeneous equation

$$AXB + CX^\star D = 0 \tag{5}$$

has only the trivial solution $X = 0$. Hence, from now on, we assume $mn = pq$ and we focus on Equation (5) instead of Equation (1).

A different reformulation of the $\star = *$ case as a linear system (in the homogeneous case) is the following.

Lemma 4. *Equation* (5)*, with $\star = *$, has a unique solution if and only if the linear system of equations*

$$\begin{aligned} AXB + CYD &= 0, \\ D^*XC^* + B^*YA^* &= 0, \end{aligned} \tag{6}$$

has a unique solution.

Proof. Let us first assume that (5) has a nonzero solution X. Then this gives a nonzero solution (X, X^*) of (6).

To prove the converse, let (X, Y) be a nonzero solution of (6). Then, the matrix $X + Y^*$ is a solution of (5). If $X + Y^*$ is zero, then $Y = -X^*$, and in this case $\mathfrak{i}X$ is a nonzero solution of (5), with $\mathfrak{i} := \sqrt{-1}$. □

The matrix associated to (6) after applying the vec operator is

$$M = \begin{bmatrix} B^\top \otimes A & D^\top \otimes C \\ \overline{C} \otimes D^* & \overline{A} \otimes B^* \end{bmatrix}. \tag{7}$$

2 Proof of Theorem 3

Here we provide a proof of Theorem 3, which gives a complete characterization of the uniqueness of solution of (1) for any right-hand side. We consider separately the case in which every coefficient is non-square, namely $p \notin \{m, n\}$, which is treated in Section 2.1, and the case in which two coefficients are square and two are nonsquare, namely $p \in \{m, n\}$, which is treated in Section 2.2. The case where all coefficients are square is Theorem 2.

2.1 The case $p \notin \{m, n\}$

In this section we show that Theorem 3 holds if $p \notin \{m, n\}$. Note that, because of the restriction $mn = pq$, this also implies that $q \notin \{m, n\}$, so this situation covers all instances of Theorem 3 where none of the coefficient matrices are square.

Lemma 5. *Let $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$. If $mn = pq$ and $p \notin \{m, n\}$ then $AXB + CX^\star D = 0$ has a nonzero solution.*

Proof. We consider separately four cases, depending on whether p is smaller or larger than m and n.

1. $p < \min\{m, n\}$. There are two nonzero vectors u, v such that $Au = 0$ and $Cv = 0$, because of the dimensions of these two matrices. Then $X = uv^\star$ is a nonzero solution of (5).
2. If $p > \max\{m, n\}$, the identity $mn = pq$ implies $q < \min\{m, n\}$. Then, there are two nonzero vectors u, v such that $v^\star B = 0$, $u^\star D = 0$, and $X = uv^\star$ is a nonzero solution of (5).
3. $m < p < n$. In this case, and because of the identity $mn = pq$, we have $m < q < n$ as well. Therefore, $m < \min\{p, q\}$. In particular, there exist nonzero vectors u, v such that $u^\top A = 0$, $v^\top D^\top = 0$.

   Now we consider the cases:

   (a) $\star = \top$. As argued in Section 1.2, Equation (5) is equivalent to the linear system $M \operatorname{vec} X = 0$, with the matrix $M \in \mathbb{C}^{(mn)\times(mn)}$ as in (4). Then, $(v^\top \otimes u^\top)M = 0$, so M is singular and (5) has a nonzero solution.

   (b) $\star = *$. As a consequence of Lemma 4, Equation (5) has a nonzero solution if and only if the (square) matrix (7) is singular. It is easy to verify that $\begin{bmatrix} v^\top \otimes u^\top & u^* \otimes v^* \end{bmatrix} M = 0$, so M is indeed singular.
4. $n < p < m$. By setting $Y = X^\star$, Equation (5) is equivalent to $CYD + AY^\star B = 0$, so we use the result for the previous case on this equation. □

2.2 The case $p \in \{m, n\}$

We have seen in Lemma 5 that if all coefficient matrices in (1) are rectangular, then (5) cannot have a unique solution when $mn = pq$. In order for (1) to have a unique solution, for any right-hand side E, then (5) must have a unique solution. Therefore, it remains to consider the cases $p = m$ and $p = n$. For the reader's convenience, we include a complete proof of Theorem 3.

Proof of Theorem 3. We first show that we can restrict ourselves to the case $mn = pq$. In order for (1) to have a unique solution, for any right-hand side E, the coefficient matrix associated with Equation (1) must be square, and this implies $mn = pq$, as explained in Section 1.2. On the other hand, the conditions in each case (1)–(3) in the statement, imply $mn = pq$.

Secondly, we observe that we can consider just the case $E = 0$, since for $mn = pq$ the square matrix associated with Equation (1) is square, and the unique solvabilty of the generalized $\star$-Sylvester equation (1) is equivalent to the existence of a unique solution of the homogeneous equation (5).

We continue by considering the case $p \notin \{m, n\}$. As a direct consequence of Lemma 5, the statement is true in this case. More precisely, the solution of (5) is non-unique, and $\mathcal{Q}(\lambda)$ is singular because it is non-square. To see this, note that $\mathcal{Q}(\lambda)$ has size $(p+q)\times(m+n)$. If $p+q=m+n$ this fact, together with the identity $mn=pq$, would imply $\{m,n\}=\{p,q\}$, since both m,n and p,q are the roots of the same quadratic polynomial, namely $x^2-(m+n)x+mn$.

It remains to consider the cases where either $p=m$ or $p=n$, which imply $q=n$ and $q=m$, respectively, due to the constraint $mn=pq$. Let us assume that $p=m>n=q$, so that D has more rows than columns, and there is some $u\neq 0$ such that $u^\star D=0$. If B is singular, then there is some $v\neq 0$ such that $v^\star B=0$. Therefore $X=uv^\star$ is a nontrivial solution of (5).

Assume now that (5) has a unique solution. Then, B is guaranteed to be nonsingular, and $AXB+CX^\star D=0$ has a unique solution if and only if $AX+CX^\star DB^{-1}=0$ has a unique solution. Moreover, we can find an invertible matrix $Q\in\mathbb{C}^{m\times m}$ such that

$$QDB^{-1}=\begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \tag{8}$$

with $D_1\in\mathbb{C}^{n\times n}$. This allows us to rewrite (5), after multiplying on the right by B^{-1}, and setting $Y=Q^{-\star}X$, in the equivalent form

$$AQ^\star\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}+C\begin{bmatrix} Y_1^\star & Y_2^\star \end{bmatrix}\begin{bmatrix} D_1 \\ 0 \end{bmatrix}=0, \tag{9}$$

where Y_1 has size $n\times n$ and Y_2 has size $(m-n)\times n$. If we partition $AQ^\star$ conformally as

$$AQ^\star=\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \tag{10}$$

with $\widetilde{A}_{11}\in\mathbb{C}^{n\times n},\widetilde{A}_{22}\in\mathbb{C}^{(m-n)\times(m-n)}$, then the block $\left[\begin{smallmatrix} \widetilde{A}_{12} \\ \widetilde{A}_{22} \end{smallmatrix}\right]$ has full column rank. If that were not the case we could find $Y_2\neq 0$ such that

$$\begin{bmatrix} \widetilde{A}_{12} \\ \widetilde{A}_{22} \end{bmatrix}Y_2=0,$$

and this would imply

$$AQ^\star\begin{bmatrix} 0 \\ Y_2 \end{bmatrix}+C\begin{bmatrix} 0 & Y_2^\star \end{bmatrix}\begin{bmatrix} D_1 \\ 0 \end{bmatrix}=0,$$

so equation (9) would have a nontrivial solution. Then, there is an invertible matrix $U\in\mathbb{C}^{m\times m}$ such that

$$UAQ^\star=U\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}=\begin{bmatrix} \widehat{A}_{11} & 0 \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix},$$

with $\widehat{A}_{22}\in\mathbb{C}^{(m-n)\times(m-n)}$ nonsingular. If we set $UC=\left[\begin{smallmatrix} \widehat{C}_1 \\ \widehat{C}_2 \end{smallmatrix}\right]$, with $\widehat{C}_1\in\mathbb{C}^{n\times n},\widehat{C}_2\in\mathbb{C}^{(m-n)\times n}$ then, after multiplying on the left by U, (9) is equivalent to the system

$$\begin{array}{rcl} \widehat{A}_{11}Y_1 + \widehat{C}_1 Y_1^\star D_1 & = & 0, \\ \widehat{A}_{22}Y_2 & = & -(\widehat{A}_{21}Y_1 + \widehat{C}_2 Y_1^\star D_1). \end{array}$$

Since $\widehat{A}_{22}$ is nonsingular, the above system has a unique solution if and only if the first equation

$$\widehat{A}_{11}Y_1 + \widehat{C}_1 Y_1^\star D_1 = 0 \tag{11}$$

has a unique solution.

We are now ready to relate the uniqueness of solution of (5) to the spectral properties of the pencil $\mathcal{Q}(\lambda)$ in the statement. We perform the following left and right invertible transformations to $\mathcal{Q}(\lambda)$:

$$\begin{bmatrix} B^{-\star} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \begin{bmatrix} Q^\star & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \lambda D_1^\star & 0 & I \\ \widehat{A}_{11} & 0 & \lambda \widehat{C}_1 \\ \widehat{A}_{21} & \widehat{A}_{22} & \lambda \widehat{C}_2 \end{bmatrix}. \tag{12}$$

Set

$$\widehat{\mathcal{Q}}(\lambda) = \begin{bmatrix} \lambda D_1^\star & I \\ \widehat{A}_{11} & \lambda \widehat{C}_1 \end{bmatrix}.$$

Then, by (12), $\det \mathcal{Q}(\lambda) = \alpha \det \widehat{\mathcal{Q}}(\lambda)$, where $\alpha = \pm(\det \widehat{A}_{22} \det B^\star)/(\det U \det Q^\star)$ is a nonzero constant.

Since all coefficient matrices in (11) are square and with the same size, namely $n \times n$, Theorem 2 implies that $\widehat{\mathcal{Q}}(\lambda)$ is regular (so $\mathcal{Q}(\lambda)$ is regular as well) and

- If $\star = \top$, $\Lambda(\widehat{\mathcal{Q}}) \setminus \{\pm 1\}$ is reciprocal free and $m_1(\widehat{\mathcal{Q}}) = m_{-1}(\widehat{\mathcal{Q}}) \leqslant 1$.
- If $\star = *$, $\Lambda(\widehat{\mathcal{Q}})$ is $*$-reciprocal free.

Note that (12) implies that $\widehat{\Lambda}(\mathcal{Q}) = \Lambda(\widehat{\mathcal{Q}})$, since $\widehat{Q}$ is obtained by deflating $m - n$ infinite eigenvalues from the pencil in the right hand side of (12). So the previous two conditions are equivalent to the conditions on the spectrum of $\mathcal{Q}(\lambda)$ in the statement.

To prove the converse, let us assume that B is invertible, and that $\mathcal{Q}(\lambda)$ is regular and its spectrum satisfies the conditions in the statement. Then we can define the matrix Q as in (8) and we arrive to (10). Again, the block $\begin{bmatrix} \widetilde{A}_{12} \\ \widetilde{A}_{22} \end{bmatrix}$ has full column rank since, otherwise, the pencil $\mathcal{Q}(\lambda)$ would be singular. This is an immediate consequence of the identity:

$$\begin{bmatrix} B^{-\star} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \begin{bmatrix} Q^\star & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \lambda D_1^\star & 0 & I \\ \widetilde{A}_{11} & \widetilde{A}_{12} & \lambda \widehat{C}_1 \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \lambda \widehat{C}_2 \end{bmatrix},$$

where $\begin{bmatrix} \widehat{C}_1 \\ \widehat{C}_2 \end{bmatrix} = UC$.

Proceeding as before, we conclude, that (5) is equivalent to (11). Using the fact that $\widehat{\Lambda}(\mathcal{Q}) = \Lambda(\widehat{\mathcal{Q}})$ and applying Theorem 2, with the hypotheses on $\mathcal{Q}(\lambda)$, to Equation (11), we conclude that the latter has a unique solution, and this implies that (5) has a unique solution.

Now, let us assume that $m < n$. After applying the $\star$ operator in (5) and setting $Y = X^\star$, we arrive to the equivalent equation $B^\star Y A^\star + D^\star Y^\star C^\star = 0$. This equation is of the form (5), with the coefficients of the first summand being square, $B^\star \in \mathbb{C}^{n\times n}$, $A^\star \in \mathbb{C}^{m\times m}$ and $n > m$, so we are in the same conditions as before. Applying the result just proved for this case, we get that the unique solvability is equivalent to requiring that A is invertible and that the pencil

$$\widetilde{\mathcal{Q}}(\lambda) = \begin{bmatrix} \lambda C & A \\ B^\star & \lambda D^\star \end{bmatrix}$$

satisfies the conditions in the statement. But, since

$$\widetilde{\mathcal{Q}}(\lambda) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{Q}(\lambda) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

this is equivalent to requiring that $\mathcal{Q}(\lambda)$ satisfies these conditions as well.

For the case $p = n \neq m$, we apply the $\star$ operator in (5), and we arrive to the equivalent equation $D^\star X C^\star + B^\star X^\star A^\star = 0$, whose coefficients are in the conditions of the previous case. The pencil associated to this last equation is

$$\widehat{\mathcal{Q}}(\lambda) = \begin{bmatrix} \lambda A & C \\ D^\star & \lambda B^\star \end{bmatrix}.$$

This pencil is the reversal of the pencil:

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{Q}(\lambda),$$

so $\Lambda(\widehat{\mathcal{Q}}) = \Lambda^{-1}(\mathcal{Q}) := \{\lambda^{-1} : \lambda \in \Lambda(\mathcal{Q})\}$, including multiplicities. In particular, the conditions on being $(*)$-reciprocal free in the statement are the same for both pencils, and the roles of the zero and the infinite eigenvalue are exchanged.

Finally, the case $p = m = n$ is Theorem 2, proved in [6]. $\square$

Remark 6. The conditions $n = q$ in part 1, and $m = q$ in part 2 in Theorem 3 are redundant, but we have included them for emphasis. These conditions are a consequence of the fact that $\mathcal{Q}(\lambda)$ in (3) is regular and the other conditions on the size, namely $p = m$ and $p = n$, respectively. As indicated in the proof of Theorem 3, since $\mathcal{Q}(\lambda)$ has size $(p+q) \times (m+n)$, if it is regular, it must be, in particular, square, and this implies $m + n = p + q$.

2.3 Necessity of the invertibility conditions

The characterization for the uniqueness of solution of (1) in Theorem 3 involves, in cases 1 and 2, the invertibility of some of the coefficient matrices. One might wonder if these conditions are really needed, or whether they could be stated in terms of spectral properties of the pencil $\mathcal{Q}(\lambda)$. However, the following example shows that the uniqueness of solution does not depend solely on the eigenvalues of $\mathcal{Q}(\lambda)$. Consider the following generalized $\top$-Sylvester equations (the same example works for the $\star = *$ case):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0, \tag{13}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0. \tag{14}$$

The above equations have associated pencils defined as follows:

$$\mathcal{Q}_1(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 0 \\ \hline 1 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right], \qquad \mathcal{Q}_2(\lambda) = \left[\begin{array}{cc|c} \lambda & 0 & 1 \\ \hline 0 & 0 & \lambda \\ 0 & 1 & 0 \end{array}\right].$$

The above pencils are the same up to row and column permutations, so they have the same eigenvalues and Kronecker form. However, the corresponding generalized Sylvester equations (13)–(14) can be rewritten, respectively, as

$$x = 0, \qquad \text{and} \qquad x = y = 0.$$

Then (13) has infinitely many solutions, while (14) has a unique solution.

3 Some corollaries

The characterization given in Theorem 3 depends on spectral properties of the pencil $\mathcal{Q}(\lambda)$ in (3), which has twice the size of the coefficient matrices of Equation (1). With some additional effort, we can provide a characterization in terms of pencils with exactly the same size.

Corollary 7. *Let* $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}$, *and* $D \in \mathbb{C}^{m\times q}$. *Then the equation* $AXB + CX^\star D = E$ *has a unique solution, for any right-hand side* E, *if and only if one of the following situations holds:*

(1) $p = m \leqslant n = q$, *A is invertible, the pencil* $\mathcal{P}_1(\lambda) := B^\star - \lambda D^\star A^{-1} C$ *is regular and*

- *If* $\star = \top$, $\Lambda(\mathcal{P}_1) \setminus \{1\}$ *is reciprocal free and* $m_1(\mathcal{P}_1) \leqslant 1$.
- *If* $\star = *$, $\Lambda(\mathcal{P}_1)$ *is $*$-reciprocal free.*

(2) $p = m \geqslant n = q$, *B is invertible, the pencil* $\mathcal{P}_2(\lambda) := A^\star - \lambda D B^{-1} C^\star$ *is regular and*

- *If* $\star = \top$, $\Lambda(\mathcal{P}_2) \setminus \{1\}$ *is reciprocal free and* $m_1(\mathcal{P}_2) \leqslant 1$.
- *If* $\star = *$, $\Lambda(\mathcal{P}_2)$ *is $*$-reciprocal free.*

(3) *$p = n \leqslant m = q$, C is invertible, the pencil $\mathcal{P}_3(\lambda) := D^\star - \lambda B^\star C^{-1} A$ is regular and*

   - *If $\star = \top$, $\Lambda(\mathcal{P}_3) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_3) \leqslant 1$.*
   - *If $\star = *$, $\Lambda(\mathcal{P}_3)$ is $*$-reciprocal free.*

(4) *$p = n \geqslant m = q$, D is invertible, the pencil $\mathcal{P}_4(\lambda) := C^\star - \lambda B D^{-1} A^\star$ is regular and*

   - *If $\star = \top$, $\Lambda(\mathcal{P}_4) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}_4) \leqslant 1$.*
   - *If $\star = *$, $\Lambda(\mathcal{P}_4)$ is $*$-reciprocal free.*

Proof. Let us assume first that (1) has a unique solution, for any right-hand side E. Then Theorem 3 implies that at least one of the following situations holds: (C1) $p = m < n = q$ and A is invertible, (C2) $p = m > n = q$ and B is invertible, (C3) $p = n < m = q$ and D is invertible, (C4) $p = n > m = q$ and C is invertible, or (C5) $p = m = n = q$ and at least one of A, B, C, D is invertible. Let us first assume that case (C1) holds. We can perform the following unimodular equivalence on $\mathcal{Q}(\lambda)$ (which preserves the finite spectrum):

$$\begin{bmatrix} I & -\lambda D^\star A^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} = \begin{bmatrix} 0 & B^\star - \lambda^2 D^\star A^{-1} C \\ A & \lambda C \end{bmatrix}. \tag{15}$$

This shows, in particular, that $\mathcal{Q}(\lambda)$ is regular if and only if $\mathcal{P}_1(\lambda)$ is regular. Moreover, we can also perform the following strict equivalence transformation to $\operatorname{rev} \mathcal{Q}(\lambda)$:

$$\begin{bmatrix} D^\star & \lambda B^\star \\ \lambda A & C \end{bmatrix} \begin{bmatrix} A^{-1} C & A^{-1} \\ -\lambda I & 0 \end{bmatrix} = \begin{bmatrix} D^\star A^{-1} C - \lambda^2 B^\star & D^\star A^{-1} \\ 0 & \lambda I \end{bmatrix}. \tag{16}$$

Taking determinants in (16) we arrive to

$$\det(\operatorname{rev} \mathcal{Q}(\lambda)) = \pm \lambda^{m-n} \det(A^{-1}) \det(\operatorname{rev} \mathcal{P}_1(\lambda)). \tag{17}$$

Equations (15) and (17) show that $\widehat{\Lambda}(\mathcal{Q}) = \sqrt{\Lambda(\mathcal{P}_1)} := \{\mu : \ \mu^2 \in \Lambda(\mathcal{P}_1)\}$, including multiplicities. Then Theorem 3 implies that claim 1 in the statement holds.

If case (C2) holds, then we apply the $\star$ operator in (1) and apply the previous arguments to the new equation and its corresponding pencil.

If case (C3) holds, then after introducing the change of variables $Y = X^\star$, the roles of A, B and C, D are exchanged, so we apply the same arguments as in case (C1) to the corresponding pencil, $\mathcal{P}_3(\lambda)$.

In case (C4), we apply the $\star$ operator in (1) and introduce the change of variables $Y = X^\star$. Then we apply the same arguments as for case (C1) to the pencil corresponding to this new equation.

Finally, if we are in case (C5), at least one of 1–4 in the statement holds, and we are done.

To prove the converse, let us assume that any of 1–4 in the statement holds. Then, reversing the previous arguments, we can conclude that at least one of the situations 1–3 in the statement of Theorem 3 occurs, and Theorem 3 implies that (1) has a unique solution, for any right-hand side. ☐

As another corollary of Theorem 3 we get an extension of [3, Lemma 5.10] and [14, Lemma 8] for the $\star$-Sylvester equation $AX + X^\star D = E$ (see also [5, Th. 10, Th. 11]) to the case of rectangular coefficients.

Corollary 8. *Let $A \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{n\times m}$. Then the equation $AX + X^\star D = E$ has a unique solution, for any right-hand side E, if and only if the matrix pencil $\mathcal{P}(\lambda) = A - \lambda D^\star$ is regular and:*

- *If $\star = \top$, $\Lambda(\mathcal{P}) \setminus \{1\}$ is reciprocal free and $m_1(\mathcal{P}) \leqslant 1$.*
- *If $\star = *$, $\Lambda(\mathcal{P})$ is $*$-reciprocal free.*

4 Explicit characterization for the generalized Sylvester equation

In this section, we provide an explicit solution of problems **SR**, **OR**, and **UR** for Equation (2). These characterizations follow from the results and lemmas in [11], but we state them explicitly in Theorem 9. Unlike the characterization given in Theorem 3 of **UR** for Equation (1), the characterizations for Equation (2) depend on further constraints on the Kronecker canonical form (KCF) of the matrix pencils $A - \lambda C$ and $D^\top - \lambda B^\top$, and not just on their spectrum. Though the KCF is a standard canonical form that can be found in most of the basic references on matrix pencils, we refer the reader to [5, Th. 2], since we follow the notation in that paper. In particular, $J(\alpha)$ denotes a *Jordan block* associated with the eigenvalue α, including $\alpha = \infty$ (which is denoted by N in [5]), L_ε denotes a *right singular block* of size $\varepsilon \times (\varepsilon + 1)$, and $L_\eta^\top$ denotes a *left singular block* of size $(\eta + 1) \times \eta$.

Theorem 9. *Let $A, C \in \mathbb{C}^{p\times m}$ and $B, D \in \mathbb{C}^{n\times q}$.*

SR *Equation* (2) *has* at least *one solution, for any right-hand side E, if and only if the following two conditions are satisfied:*

- *the (possibly singular) pencils $A-\lambda C$ and $D^\top - \lambda B^\top$ have no common eigenvalues,* and
- *if the KCF of either $A - \lambda C$ or $D^\top - \lambda B^\top$ contains one block $L_\eta^\top$, then the KCF of the other pencil is a direct sum of blocks L_{ε_i} with $\varepsilon_i \leqslant \eta$.*

OR *Equation* (2) *has* at most *one solution, for any right-hand side E, if and only if the following two conditions are satisfied:*

- *the (possibly singular) pencils* $A-\lambda C$ *and* $D^\top - \lambda B^\top$ *have no common eigenvalues,* and
- *if the KCF of either* $A-\lambda C$ *or* $D^\top - \lambda B^\top$ *contains one block* L_ε*, then the KCF of the other pencil is a direct sum of blocks* $L_{\eta_i}^\top$ *with* $\eta_i \leqslant \varepsilon$*.*

UR *Equation* (2) *has* exactly *one solution, for any right-hand side* E*, if and only if one of the following situations hold:*

- *the pencils* $A-\lambda C$ *and* $D^\top - \lambda B^\top$ *are regular and have no common eigenvalues,* or
- *there is some* $s \in \mathbb{Z}^+$ *such that the KCF of either* $A-\lambda C$ *or* $B^\top - \lambda D^\top$ *is a direct sum of blocks* L_s*, and the KCF of the other pencil is a direct sum of blocks* $L_s^\top$*.*

Proof. Let $P_1(A-\lambda C)Q_1 = \widehat{A} - \lambda \widehat{C}$ and $P_2^\top(D^\top - \lambda B^\top)Q_2^\top = \widehat{D}^\top - \lambda \widehat{B}^\top$ be the KCFs of $A-\lambda C$ and $D^\top - \lambda B^\top$, respectively. Equation (2) is equivalent to

$$\widehat{A}\widehat{X}\widehat{B} - \widehat{C}\widehat{X}\widehat{D} = \widehat{E}, \quad \text{with } \widehat{X} = Q_1^* X Q_2, \quad \widehat{E} = P_1 E P_2. \tag{18}$$

Partitioning the matrices conformably with the (possibly rectangular) blocks in the KCFs, we get

$$\begin{bmatrix} \widehat{A}_{11} & & & \\ & \widehat{A}_{22} & & \\ & & \ddots & \\ & & & \widehat{A}_{pp} \end{bmatrix} \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} & \dots & \widehat{X}_{1q} \\ \widehat{X}_{21} & \widehat{X}_{22} & \dots & \widehat{X}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{X}_{p1} & \widehat{X}_{p2} & \dots & \widehat{X}_{pq} \end{bmatrix} \begin{bmatrix} \widehat{B}_{11} & & & \\ & \widehat{B}_{22} & & \\ & & \ddots & \\ & & & \widehat{B}_{qq} \end{bmatrix}$$
$$- \begin{bmatrix} \widehat{C}_{11} & & & \\ & \widehat{C}_{22} & & \\ & & \ddots & \\ & & & \widehat{C}_{pp} \end{bmatrix} \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} & \dots & \widehat{X}_{1q} \\ \widehat{X}_{21} & \widehat{X}_{22} & \dots & \widehat{X}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{X}_{p1} & \widehat{X}_{p2} & \dots & \widehat{X}_{pq} \end{bmatrix} \begin{bmatrix} \widehat{D}_{11} & & & \\ & \widehat{D}_{22} & & \\ & & \ddots & \\ & & & \widehat{D}_{qq} \end{bmatrix}$$
$$= \begin{bmatrix} \widehat{E}_{11} & \widehat{E}_{12} & \dots & \widehat{E}_{1q} \\ \widehat{E}_{21} & \widehat{E}_{22} & \dots & \widehat{E}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{E}_{p1} & \widehat{E}_{p2} & \dots & \widehat{E}_{pq} \end{bmatrix},$$

and (18) is equivalent to the system of pq independent equations

$$\widehat{A}_{ii}\widehat{X}_{ij}\widehat{B}_{jj} - \widehat{C}_{ii}\widehat{X}_{ij}\widehat{D}_{jj} = \widehat{E}_{ij}, \quad i = 1, 2, \dots, p,\ j = 1, 2, \dots, q. \tag{19}$$

In particular, the existence (resp. uniqueness) of solution X of (2), for any right-hand side E, is equivalent to the simultaneous existence (resp. uniqueness) of the solution $\widehat{X}_{ij}$ of (19), for any right-hand side $\widehat{E}_{ij}$, and for any $i = 1, 2, \dots, p, j = 1, 2, \dots, q$.

$A-\lambda C$	$D^\top - \lambda B^\top$	At least one solution?	At most one solution?
$J(\alpha)$	$J(\alpha)$	No	No
$J(\alpha)$	$J(\beta)$ $(\beta \neq \alpha)$	Yes	Yes
$L_\eta^\top$	L_ε	Only when $\eta \geqslant \varepsilon$	Only when $\eta \leqslant \varepsilon$
L_ε	$L_\eta^\top$	Only when $\eta \geqslant \varepsilon$	Only when $\eta \leqslant \varepsilon$
$L_{\eta_1}^\top$	$L_{\eta_2}^\top$	No	Yes
L_{ε_1}	L_{ε_2}	Yes	No
$J(\alpha)$	L_ε	Yes	No
$J(\alpha)$	$L_\eta^\top$	No	Yes
$L_\eta^\top$	$J(\alpha)$	No	Yes
L_ε	$J(\alpha)$	Yes	No

Table 2: Summary of the results on existence and uniqueness of solutions for the equation $AXB - CXD = E$ when $A - \lambda C$ and $D^\top - \lambda B^\top$ are Kronecker blocks, obtained from the lemmas in [11, Sec. 4 and 5]. The Jordan blocks include $\alpha = \infty$.

Solvability and uniqueness conditions for the systems (19), where $\widehat{A}_{ii} - \lambda \widehat{C}_{ii}$ and $\widehat{D}_{jj}^\top - \lambda \widehat{B}_{jj}^\top$ are blocks from the KCF, are provided in [11]. We recall them in Table 2. One can check, using this table, that each equation of the system (19) has at least one solution if and only if the **SR** conditions in the statement of the theorem hold; similarly, each equation has at most one solution if and only if the **OR** conditions hold, and it has exactly one solution if and only if the **UR** conditions hold. ☐

4.1 An alternative characterization for UR

The criterion for unique solvability for each right-hand side (**UR**) for the generalized Sylvester equation (2) gives a peculiar restriction that can be expressed in terms of the ratios between the two dimensions of the involved pencils. We denote by $\mathbb{Z}^+$ the set of positive integers.

Corollary 10. *Let $A, C \in \mathbb{C}^{p\times m}$ and $B, D \in \mathbb{C}^{n\times q}$. If the generalized Sylvester equation (2) has a unique solution for any right-hand side, then $p/m = n/q = d$, for some*

$$d \in \{1\} \cup \left\{ \frac{s}{s+1} : s \in \mathbb{Z}^+ \right\} \cup \left\{ \frac{s+1}{s} : s \in \mathbb{Z}^+ \right\}.$$

The necessary condition given in Corollary 10 is not sufficient. In order to give a sufficient condition, we need the following result.

Lemma 11. *Let $A, C \in \mathbb{R}^{p\times m}$, and $\varepsilon \in \mathbb{Z}^+$. Then, the KCF of the pencil $A - \lambda C$ consists only of of blocks of the form L_ε if and only if the $p(\varepsilon + 1) \times m\varepsilon$*

matrix

$$M_\varepsilon(A,C) := \begin{bmatrix} A & & & \\ C & A & & \\ & C & \ddots & \\ & & \ddots & A \\ & & & C \end{bmatrix}$$

is square and invertible. Similarly, the KCF of $A - \lambda C$ is composed uniquely of blocks of the form $L_\eta^\top$ if and only if $M_\eta(A^\top, C^\top)$ is square and invertible.

Proof. Let us first suppose that $M_\varepsilon(A,C)$ is square and invertible. Let $p_1 \times m_1, p_2 \times m_2, \ldots, p_k \times m_k$ be the sizes of the blocks in the KCF of $A - \lambda C$. It follows from the arguments in [8, Section XII.3] that $M_\varepsilon(A,C)$ has a nontrivial kernel if and only if there is a polynomial vector $x(\lambda)$ of degree strictly smaller than ε such that $(A - \lambda C)x(\lambda) = 0$, or, equivalently, if and only if the KCF of $A - \lambda C$ contains a block L_{ε_2} with $\varepsilon_2 < \varepsilon$. Hence, if $M_\varepsilon(A,C)$ is invertible, then the KCF of $A - \lambda C$ contains only Jordan blocks, together with blocks $L_\eta^\top$, and blocks L_{ε_2} with $\varepsilon_2 \geqslant \varepsilon$. For each of these blocks, one can check that $\frac{m_i}{p_i} \leqslant \frac{\varepsilon+1}{\varepsilon}$, for $i = 1, \ldots, k$, and the equality holds only for blocks of type L_ε. In particular, we have

$$p(\varepsilon+1) = \sum_{i=1}^{k} p_i(\varepsilon+1) \geqslant \sum_{i=1}^{k} m_i \varepsilon = m\varepsilon.$$

Since $M_\varepsilon(A,C)$ is square, equality must hold for all $i = 1, \ldots, k$, which means that all blocks are of type L_ε.

Now let us prove the other direction: suppose that the KCF of $A - \lambda C$ consists only of blocks L_ε. Then, $p = k\varepsilon$ and $m = k(\varepsilon+1)$, so $p(\varepsilon+1) = m\varepsilon$, and $M_\varepsilon(A,C)$ is square. Moreover, again by the same reasonings in [8, Section XII.3] as above, $M_\varepsilon(A,C)$ has trivial kernel, so it is invertible.

The second claim in the statement follows by applying the first statement to $A^\top - \lambda C^\top$. ☐

Using Lemma 11 we can give an alternative version of the **UR** characterization in Theorem 9 that does not involve the KCF explicitly.

Theorem 12. *Let $A, C \in \mathbb{C}^{p\times m}$ and $B, D \in \mathbb{C}^{n\times q}$. Equation* (2) *has* exactly *one solution, for any right-hand side E, if and only if one of the following situations hold:*

- *$p = m, q = n$, the pencils $A - \lambda C$ and $D^\top - \lambda B^\top$ are regular and have no common eigenvalues,* or
- *$p < m, n < q$, $s = p/(m-p) = n/(q-n)$ is a positive integer, and the square matrices $M_s(A,C)$ and $M_s(B,D)$ are both invertible,* or
- *$p > m, n > q$, $s = m/(p-m) = q/(n-q)$ is a positive integer, and the square matrices $M_s(A^\top, C^\top)$ and $M_s(B^\top, D^\top)$ are both invertible.*

Proof. The proof is an immediate consequence of Theorem 9, part **UR**, and Lemma 11, just taking into account that if the KCF of an $m \times n$ pencil (respectively, the KCF of an $n \times m$ pencil) is a direct sum of k blocks L_s (resp., $L_s^\top$), for some fixed s, then, as we have seen in the proof of Lemma 11, it must be $m = ks, n = k(s+1)$ (resp., $m = k(s+1), n = ks$), which implies $n > m$ and $s = m/(n-m)$ (resp., $n < m$ and $s = n/(m-n)$). $\square$

5 Conclusions and open problems

We have provided necessary and sufficient conditions for the generalized $\star$-Sylvester equation (1) to have a unique solution for any right-hand side E (**UR**). In particular, the coefficient matrix of the associated linear system must be square, which is equivalent to the condition $mn = pq$, and the problem becomes equivalent to characterizing the uniqueness of solution of the homogeneous equation (5). The characterization that we have obtained extends the recent one in [6] for the case of square coefficients. We have also reviewed the solution of problems **SR** (solvability for any right-hand side), **OR** (at most one solution for any right-hand side), and **UR** (unique solvability for any right-hand side) for the generalized Sylvester equation (2).

It is interesting to compare the conditions for unique solvability (**UR**) for the two equations (1) and (2), given in Theorems 3 and 9 since, in the case of rectangular coefficients, there are more significant differences than those in the case of square coefficients. For the generalized Sylvester equation (2), the only additional case with unique solution is when the KCF of the associated pencils $A - \lambda C$ and $D^\top - \lambda B^\top$ contains only certain singular blocks; for the generalized $\star$-Sylvester equation, the spectral properties and Kronecker invariants are not sufficient to determine the answer, and it is necessary to check the invertibility of one of the coefficients.

To our knowledge, small-pencil characterizations for questions **US**, **SR**, and **OR** for the generalized $\star$-Sylvester equation are still not present in the literature and arise as a natural open problem to approach in the future.

Acknowledgments. The authors wish to thank Emre Mengi for pointing out reference [13] and, as a consequence, references [11, 12] as well.

References

[1] M. Baumann and U. Helmke. Singular value decomposition of time-varying matrices. *Future Generation Computer Systems*, 19(3):353 – 361, 2003. Special Issue on Geometric Numerical Algorithms.

[2] H. W. Braden. The equations $A^T X \pm X^T A = B$. *SIAM J. Matrix Anal. Appl.*, 20(2):295–302, 1999.

[3] R. Byers and D. Kressner. Structured condition numbers for invariant subspaces. *SIAM J. Matrix Anal. Appl.*, 28(2):326–347, 2006.

[4] K.-W. E. Chu. The solution of the matrix equations $AXB - CXD = E$ and $(YA - DZ, YC - BZ) = (E, F)$. *Linear Algebra Appl.*, 93:93–105, 1987.

[5] F. De Terán, F. M. Dopico, N. Guillery, D. Montealegre, and N. Z. Reyes. The solution of the equation $AX + X^{\star}B = 0$. *Linear Algebra Appl.*, 438:2817–2860, 2011.

[6] F. De Terán and B. Iannazzo. Uniqueness of solution of a generalized $\star$-Sylvester matrix equation. *Linear Algebra Appl.*, 493:323–335, 2016.

[7] A. Dmytryshyn and B. Kågström. Coupled sylvester-type matrix equations and block diagonalization. *SIAM J. Matrix Anal. Appl.*, 36(2):580–593, 2016.

[8] F. R. Gantmacher. *The Theory of Matrices. Vols. 1, 2.* Chelsea Publishing Co., New York, 1959.

[9] R. Granat, B. Kågström, and D. Kressner. Matlab tools for solving periodic eigenvalue problems. *IFAC Proceedings Volumes*, 40(14):169–174, 2007.

[10] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, 1994.

[11] T. Košir. The matrix equation $AXD^T - BXC^T = E$. Technical Report (Research paper 737), Dept. of Mathematics and Statistics, Univ. of Calgary, 1992.

[12] T. Košir. Kronecker bases for linear matrix equations, with application to two-parameter eigenvalue problems. *Linear Algebra Appl.*, 249:259–288, 1996.

[13] D. Kressner, E. Mengi, I. Nakić, and N. Truhar. Generalized eigenvalue problems with specified eigenvalues. *IMA J. Numer. Anal.*, 34:480–501, 2014.

[14] D. Kressner, C. Schröder, and D. S. Watkins. Implicit QR algorithms for palindromic and even eigenvalue problems. *Numer. Algorithms*, 51(2):209–238, 2009.

[15] S. K. Mitra. The matrix equation $AXB + CXD = E$. *SIAM J. Appl. Math.*, 32(4):823–825, 1977.

[16] V. V. Sergeichuk. Computation of canonical matrices for chains and cycles of linear mappings. *Linear Algebra Appl.*, 376:235–263, 2004.

[17] V. Simoncini. Computational methods for linear matrix equations. *SIAM Rev.*, 58(3):377–441, 2016.

[18] Y. Yuan and H. Dai. The direct updating of damping and gyroscopic matrices. *J. Comput. Appl. Math.*, 231(1):255–261, 2009.