# Triplet representations for matrix equations in queuing theory

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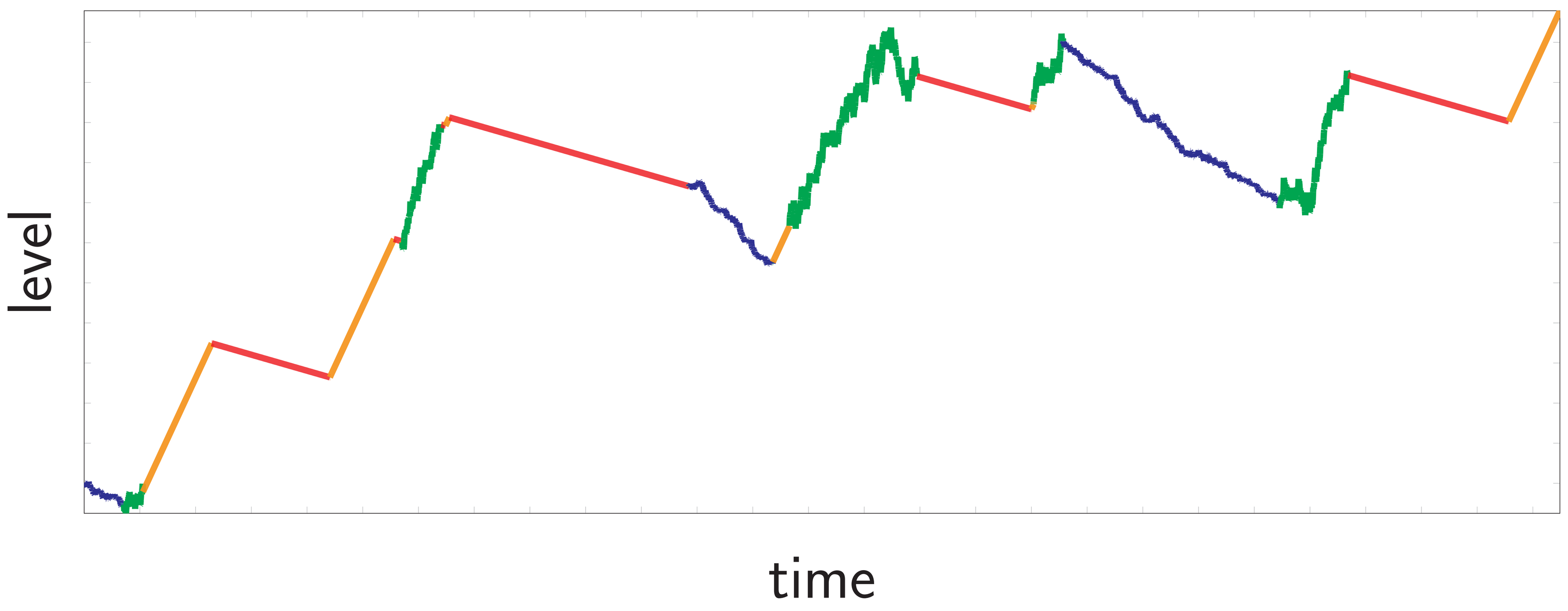
## Our problem

Markov-modulated fluid queues and Brownian motion: liquid in an infinite buffer; in- and out-flow (and possibly BM variance) depend on environment state (continuous-time Markov chain)




Stationary measure for level $x$: $p(x) \in \mathbb{R}^{1\times n}$ satisfying a BVP

$$\ddot{p}V - \dot{p}D + Q = 0 \quad + \text{boundary cond'ns at } 0 \text{ and } \infty$$

$V$: diagonal with $v_{ii} \geq 0$
$D$: diagonal with mixed-sign $d_{ii}$
$Q$: continuous-time Markov chain, $Q\mathbf{\underline{1}} = 0$, offdiag$(Q) \geqslant 0$

Key to find it: left stable invariant pair, $X^2UV - XUD + UQ = 0$
$U$ (hopefully $\geqslant 0$) "projection", $X$ square containing eigenvalues of the problem in the (open) left half-plane

## Accuracy goal

Our aim solving the problem with componentwise accuracy:

$$|M - \tilde{M}| \leqslant \varepsilon M \quad \text{for computed quantity } M \geqslant 0$$

Inequality and $|\cdot|$ to hold entrywise, even on very small entries
Problem Subtractive cancellation $\to$ loss of significant digits
Solution Avoid all the subtractions!

## Triplet representations

An M-matrix $A$ can be recovered from:

- its off-diagonal part, offdiag$(A)$, and
- $v > 0$, $w \geqslant 0$ such that $Av = w$

**Example 1**

$$\begin{bmatrix} 1 & -1 \\ -1 & 1+\varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$

Values in red (offdiag$(A)$, $v$, $w$) uniquely determine the matrix

With a triplet representation, one can run subtraction-free Gaussian elimination on $A$ (*GTH-like algorithm*): solves systems with componentwise error $O(n^3\mathbf{u})$. No condition number!

Matrix entries —ill-conditioned, $\kappa_{cw}(A)\mathbf{u}$→ Triplet repres —well-conditioned, $O(n^3\mathbf{u})$→ $A^{-1}$

## The first-order case

Case $V = 0$ more studied. Formulated equivalently as nonsymmetric algebraic Riccati equation. Best class of algorithms: doubling
[Xue, Xu, Li] gave perturbation bound, highlighted need for triplets

## First-order case: doubling algorithm

Idea Compute stable modes from $\lim_{t\to\infty} \exp(At)$, via a matrix-pencil version of scaling and squaring

1. First approximation via Padé ("continuous to discrete") $\exp(tx) \approx \mathcal{C}(x) = (1+\beta x)(1-\alpha x)^{-1}$
2. Squaring, representing intermediate matrices implicitly

What we add to this case:

**Triplets**

Explicit triplets for the inversions $\implies$ subtraction-free algorithm

$$\text{triplet}(I - G_kH_k) = (\text{offdiag}(G_kH_k), \mathbf{\underline{1}}, E_k\mathbf{\underline{1}} + G_kF_k\mathbf{\underline{1}})$$
$$\text{triplet}(I - H_kG_k) = (\text{offdiag}(H_kG_k), \mathbf{\underline{1}}, F_k\mathbf{\underline{1}} + H_kE_k\mathbf{\underline{1}})$$

No need to get them back from the matrix entries

**Error analysis**

Componentwise accurate $U \geqslant 0$ and triplet representation for $X$:

$$|U - \tilde{U}| \leqslant c\mathbf{u}U, \qquad \text{same for } X$$

Coefficient $c$ grows:

- linearly with $1 - \rho(\mathcal{C}(X))$ (distance to instability)
- cubically (upper bound, linear in practice) with dimension $n$

## Second-order case: cyclic reduction

Base algorithm (from [Latouche, Nguyen]):

1. continuous-to-discrete transformation $y = \mathcal{C}(x)$
2. Cyclic reduction (CR) – classical doubling-type algorithm for quadratic problems

**Problems**

- Not subtraction-free — signs are simply wrong for that
- Infinite eigenvalues from zeros in $V$ complicate convergence

Solution: shift technique/order reduction. With correct choices and parameters, fixes both issues at the same time
Infinite eigenvalues deflated automatically

$$P(x) = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x^2 - \begin{bmatrix} * & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{bmatrix} x + \begin{bmatrix} * & + & + \\ + & * & + \\ + & + & * \end{bmatrix}$$

$$\widehat{P}(x) = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} x^2 - \begin{bmatrix} * & 0 & - \\ 0 & + & - \\ 0 & 0 & - \end{bmatrix} x + \begin{bmatrix} * & + & + \\ + & * & + \\ + & + & * \end{bmatrix}$$

$$\mathcal{C}(\widehat{P})(y) = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{bmatrix} y^2 - \begin{bmatrix} + & 0 & - \\ 0 & + & - \\ 0 & 0 & * \end{bmatrix} y + \begin{bmatrix} + & + & 0 \\ + & + & 0 \\ + & + & 0 \end{bmatrix}$$

More ingredients for a subtraction-(essentially)-free algorithm:

- Choosing parameters: $\alpha = 0$, $\beta$ from problem magnitudes

**Triplets for CR — simpler (stochastic) case**

$$\text{triplet}(\hat{B}_k) = (\text{offdiag}(\hat{B}_k), \mathbf{\underline{1}}, C_kB_kA_0\mathbf{\underline{1}})$$

- Don't compute solution $R = C_0\hat{B}_k^{-1}$ to the matrix equation, but invariant pair $U = \hat{B}_k, X = \hat{B}_k^{-1}R\hat{B}_k$

## References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- G Latouche, GT Nguyen The morphing of fluid queues into Markov-modulated Brownian motion arXiv:1311.3359
- JG Xue, SF Xu, RC Li Accurate solutions of M-matrix algebraic Riccati equations. Numer Math 2012
- GT Nguyen, F Poloni Componentwise accurate fluid queue computations using doubling algorithms In preparation